A USEFUL UNDERESTIMATE FOR THE CONVERGENCE OF INTEGRAL FUNCTIONALS.

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Abstract

This article deals with the lower compactness property of a sequence of integrands and the use of this key notion in various domains: convergence theory, optimal control, nonsmooth analysis. First about the interchange of the weak lower epi-limit and the symbol of integration for a sequence of integral functionals. These functionals are defined on a topological space $(\mathscr{X},\mathscr{T})$, where $\mathscr{X}$ is a subset of measurable functions and the $\mathscr{T}$-sequential convergence is stronger than or equal to the convergence in the Biting sense. Given a sequence $(f_n)_n$ of integrands, if the integrand f is the weak lower sequential epi-limit of the integrands f_n one of the main results of this article asserts that under the Ioffe's criterion, the $\mathscr{T}$-lower sequential epi-limit of the sequence of integral functionals at the point x is bounded below by the value of the integral functional associated to the Fenchel-Moreau biconjugate of f at the point x. Then the strong-weak semicontinuity (respectively the subdifferentiability) of integral functionals, are studied in relation with the Ioffe's criterion. This permits, with original proofs, to give new conditions for the sequential strong-weak lower semi continuity *at a given point*, and to obtain necessary and sufficient criteria for the Fréchet and the (weak) Hadamard subdifferentiability of integral functionals on general spaces, particularly on Lebesgue spaces.

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1 Introduction

The notion of convergence plays a key role in the study of variational problems and in nonlinear analysis, see [41], [42], [37], [1], [13], [15], [2], [55], [52], [50]. The convergence of a sequence of functions is often defined through the convergence of their epigraphs. Given

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a topological space $(\mathscr{X},\mathscr{T})$, one can consider on the space of subsets of $\mathscr{X}$ the Painlevé-Kuratowski convergence. This permits to define epi-convergence of a sequence of functions $(f_n)_n$ defined on $\mathscr{X}$ with extended numerical values. In fact, the use of the epigraphs of functions and the existence of a lower limit and an upper limit of a sequence of subsets allows to define the $\mathscr{T}$-upper epi-limit, $\mathscr{T}-ls_e f_n$, (respectively the $\mathscr{T}$-lower epi-limit, $\mathscr{T}-li_e f_n$) of a sequence of extended real-valued functionals. Moreover the $\mathscr{T}$-epi-limit, $\mathscr{T}-lim_e f_n$, can be defined by the coincidence of these last two epi-limits. In this case one says that the sequence $(f_n)_n$ $\mathscr{T}$-epi-converges to $\mathscr{T}-lim_e f_n$. When we consider two topologies on $\mathscr{X}$, $\mathscr{T}_1$ finer or equal than $\mathscr{T}_2$, we can define also the $(\mathscr{T}_1,\mathscr{T}_2)$ Mosco-convergence of a sequence of functionals with extended numerical values, by the coincidence of the associated $\mathscr{T}_1$ upper epi-limit and the $\mathscr{T}_2$ lower epi-limit. The case considered initially by U. Mosco is a reflexive Banach space endowed with its strong and weak topologies [41]. In [1], [13], [15], [55] the reader can find many examples and applications of the notion of epi-convergence.
For integral functionals it seems that it is of great interest not to study directly the convergence, but to make separate studies of the cases of lower and upper epi-limits. In this article, first it is considered sequential (weak) lower epi-limits of a sequence of such functionals.
Given a complete σ-finite measure space Ω, a separable Banach space E (or in some cases a reflexive separable Banach space), a topological subset $(\mathscr{X},\mathscr{T})$ of the space of classes (for almost everywhere equality) of measurable E-valued functions, and a sequence $(f_n)_n$ of measurable extended real-valued non necessarily convex integrands defined on $\Omega\times\mathrm{E}$, our first purpose is to give a " best " lower bound for the $\mathscr{T}$-sequential lower epi-limit of the associated sequence $(\mathrm{I}_{f_n})_n$ of integral functionals at a given point $x\in\mathscr{X}$. The study of convergence of integral functionals is originally started by J- L. Joly and F. De Thélin [38] in the convex case with E finite dimensional, $\mathscr{X}=\mathrm{L}_p(\Omega,\mathrm{E})$ and for Mosco-convergence; then by A. Salvadori [56] when Ω is a finite measured space, E is a reflexive separable Banach space; more recently for the slice convergence by J. Couvreux [12], Ω being a probability space and E a Banach space with separable dual, with $\mathscr{X}=\mathrm{L}_p(\Omega,\mathrm{E})$ also in the convex case. In our approach it is important to avoid any convergence, properness, or global convexity assumptions on the sequence of integrands and of functionals.
The second and third section are devoted to some known or new preliminaries on the calculus of the Fenchel-Moreau conjugate of a sequential weak lower epi-limit, on the Biting Lemma and tightness. In Section 4 a measurability property is first proved: Theorem 4.6. Then the chapter VII of C. Castaing and M. Valadier'book [9] is used and a consequence of M. Valadier results [60] section 3, [9] Theorem VII-7 on the calculus of the Fenchel-Moreau conjugate of an integral functional is put in light. Section 5 is devoted to a property of the upper epi limit $ls_e\mathrm{I}_{f_n}$ of a sequence $(\mathrm{I}_{f_n})_n$ of integral functionals defined on the dual of $\mathrm{L}_1(\Omega,\mathrm{E})$ and for the topology of uniform convergence on weakly compact sets of $\mathrm{L}_1(\Omega,\mathrm{E})$, Proposition 5.3, this result permits to give a proof of the main result of section 6 by a duality method. In Section 6, the property (P) required on the topology $\mathscr{T}$ is trivially satisfied when $\mathscr{X}=\mathrm{L}_p(\Omega,\mathrm{E})$ is endowed with a topology stronger than or equal the weak topology. Given a sequence of integrands $(f_n)_n$ defined on $\Omega\times\mathrm{E}$, let the integrand $f=seq\,\sigma-li_e f_n$ and its Fenchel-Moreau biconjugate f^{**}, the main result is Theorem 6.3; it proves that given a sequence $(x_n)_n$ $\mathscr{T}$-converging to x under mild assumptions the following inequality is true

$$\liminf_n \mathrm{I}_{f_n}(x_n)\geq \mathrm{I}_{f^{**}}(x)-\delta^+((-f_n(x_n))_n)\,,$$

where $\delta^+(.)$ is an extension to the σ-finite case of Rosenthal's modulus of uniform equi-integrability (see [26]). Following [34], we will say that a sequence $(f_n)_n$ of integrands satisfies the Ioffe's criterion at $x \in \mathscr{X}$ (with respect to $\mathscr{T}$) when the following lower compactness property holds: for every subsequence $(f_{n_k})_k$ of $(f_n)_n$ and any $\mathscr{T}$-converging sequence $(x_k)_k$ to x such that the sequence $(\mathrm{I}_{f_{n_k}}(x_k))_k$ is bounded above, the sequence of negative parts $(f^-_{n_k}(x_k))_k$ is relatively weakly compact in $\mathrm{L}_1(\Omega, \mathbb{R})$. When the Ioffe's criterion holds, the term δ^+ vanishes and the inequality announced in the abstract holds. The converse being often valid when $f(x) = f^{**}(x)$ (Theorem 6.8). In Section 7 a first application is obtained: the classical problem of strong-weak lower semicontinuity of an integral functional is considered. In this case the integrand is defined on $\Omega \times \mathrm{E} \times \mathrm{F}$, where E is a topological space and F is a separable Banach space. The global semicontinuity problem was considered by many authors, notably C. Olech [46], [47], in the case $\mathrm{L}_1(\Omega, \mathrm{E}^2)$, and completely solved by A. D. Ioffe [34] for a large class of spaces of finite dimensional-valued measurable functions. C. Castaing and P. Clauzure [10] deal, with a strengthening of Olech's techniques, the case of measurable functions infinite dimensional valued: F is a Banach space with separable dual. E. J. Balder [3], [4], [5], gives a new proof of this result using the concept of seminormality for an integral functional defined on $\mathrm{L}_1(\Omega, \mathrm{E} \times \mathrm{F})$, where F is a separable reflexive Banach space. As a significant contribution he introduces and uses the notion of Nagumo tightness. A. Bourass, B. Ferrahi and O. Kahlaoui in [7] show that the Ioffe's techniques and results can be extended to the case F is a separable reflexive Banach space. Moreover, when f is non negative, I. Fonseca and G. Leoni have weakened the assumptions on the topologies ([21] Corollary 7.9) and made a remarkable characterization of the associated relaxed energy functional [21] Theorem 7.13, but F is supposed finite dimensional. More recently C. Castaing, P.R. de Fitte, M. Valadier in their book [11] with various notions of tightness in relation with the theory of Young measures, obtain a semicontinuity result in case F is a separable Banach space, [11] Theorem 8.1.6. In this article, Theorem 7.1 is first a quantitative estimate on the lack of sequential strong-weak semicontinuity. Not only it gives other proofs of the known semicontinuity results at least when F has a strongly separable dual, [3], [4], [5], [10], [21] and [7], but in case F is reflexive, with weak assumptions on the topologies it permits to extend the Ioffe's result at *a given point* without any global convexity assumptions on the integrand, Corollary 7.6. In section 8 the lower compactness property respect to a bornology is defined. Fréchet and Hadamard lower compactness properties of a sequence of integrands are considered. Using growth conditions, very concrete examples are presented on Orlicz spaces, Propositions 8.4, 8.11 and 8.12, and on Lebesgue spaces Corollaries 8.5, 8.6, 8.8, 8.9, 8.16 and 8.17. The section 9 is first devoted to introduce the notions of Fréchet and weak Hadamard subdifferentiability. Related to the J. P Penot's characterization of Fréchet subdifferentiability of an integral functional on $\mathrm{L}_p(\Omega, \mathrm{E})$, $1 \leq p < \infty$, [51] Theorems 12 and 22, two complete characterizations for the subdifferentiability of an integral functional respect to the Fréchet and weak Hadamard bornologies, are given in a general setting: Theorem 9.2 and Theorem 9.4; they permit to reach criterions in the case of Orlicz spaces Corollary 9.5, and of Lebesgue spaces when the measure is atomless: Corollaries 9.6 and 9.7 which are equivalent to the characterizations given in [51] Theorems 12 and 22, Corollary 9.8 treats the case $p = \infty$. In section 10 some additional properties of the Fréchet subdifferentiability are reached: Theorem 10.2, Corollaries 10.11 and 10.12, they are in relation with the Fréchet lower compactness property of the differential quotients associated to the integrand (for practical examples see

the sections 8 and 9). The last section is a short study of the weak Hadamard subdifferentiability of an integral functional defined on Lebesgue spaces when the Banach space E is supposed to be reflexive separable. After a reduction with the results of the previous sections it is proved that it suffices to treat only the case $p = 1$. Recall that in this case when the measure is atomless, the Fréchet subdifferential of an integral functional coincide with the Moreau-Rockafellar subdifferential (see [51] and [14]). Corollary 9.9 gives a complete characterization of weak Hadamard subdifferentiability when the measure is atomless. The proof of Theorem 11.2 uses Theorem 6.3, and it is unreachable with the A.D Ioffe's type convexity assumptions made in the yet known results on semicontinuity.

2 Preliminaries

We adopt the following notation: $\mathbb{R}$ is the set of real numbers and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Given $(\mathscr{X}, \mathscr{T})$ a topological vector space, the set of all open neighbourhoods of x in X will be denoted by $\mathscr{N}(x)$. For a subset X of $\mathscr{X}$, the indicator function ι_X is defined by $\iota_X(x) = 0$ if $x \in X$, $+\infty$ if not. For an extended real-valued function f defined on $\mathscr{X}$ we consider its effective domain, $dom f = \{x \in \mathscr{X} : f(x) < +\infty\}$, its epigraph, $epi f = \{(x, r) \in \mathscr{X} \times \mathbb{R} : f(x) \leq r\}$ and its sublevel set of height $r \in \mathbb{R}$ $f^{\leq r} = \{x \in \mathscr{X} : f(x) \leq r\}$. The function f is said to be $\mathscr{T}$-inf-compact (respectively sequentially-$\mathscr{T}$ inf-compact) when every sublevel set is $\mathscr{T}$-compact (respectively sequentially $\mathscr{T}$-compact). When $\mathscr{X}$ is a locally convex topological space with $\mathscr{X}^*$ as topological dual we consider the duality pairing between $\mathscr{X}^*$ and $\mathscr{X}$ defined by $\langle x^*, x\rangle = x^*(x)$. The function f is said inf-compact for every slope (respectively sequentially-$\mathscr{T}$ inf-compact for every slope) if for every $x^* \in \mathscr{X}^*$ $x \mapsto f(x) - \langle x^*, x\rangle$ is an inf-compact (respectively sequentially-$\mathscr{T}$ inf-compact) function. We will say that the function f is proper if its domain is nonempty and if the function f does not take the value $-\infty$. Recall that the Fenchel-Moreau conjugate f^* is defined on $\mathscr{X}^*$ by the formula [40]:

$$f^*(x^*) = \sup_{x \in \mathscr{X}} \langle x^*, x\rangle - f(x) .$$

Definition 2.1 *Given a sequence $(\mathrm{M}_n)_n$ of subsets of $\mathscr{X}$, its $\mathscr{T}$-lower limit and $\mathscr{T}$-upper limit in the sense of Kuratowski [1], [55], [15] Definition 4.10, are defined as follows:*

$$\liminf_n \mathrm{M}_n = \{x \in \mathscr{X} : \forall \mathrm{U} \in \mathscr{N}(x), \exists m \in \mathbb{N} : \forall n \geq m,\ \mathrm{U} \cap \mathrm{M}_n \neq \emptyset\} ,$$

$$\limsup_n \mathrm{M}_n = \{x \in \mathscr{X} : \forall \mathrm{U} \in \mathscr{N}(x), \forall m \in \mathbb{N}, \exists n \geq m : \mathrm{U} \cap \mathrm{M}_n \neq \emptyset\} .$$

Remark 2.2 *Clearly if $\mathscr{X}$ is a metric space with a distance d, for a subset* M *of $\mathscr{X}$, setting $d(x, \mathrm{M}) = \inf\{d(x, m), m \in \mathrm{M}\}$ if $\mathrm{M} \neq \emptyset$, $+\infty$ if $\mathrm{M} = \emptyset$, then:*

$$\liminf_n \mathrm{M}_n = \{x \in \mathscr{X} : \lim_n d(x, \mathrm{M}_n) = 0\} , \quad \limsup_n \mathrm{M}_n = \{x \in \mathscr{X} : \liminf_n d(x, \mathrm{M}_n) = 0\}.$$

Definition 2.3 *([1], [15]) Given a sequence $(f_n)_n$ of extended real-valued functions, the upper epi-limit (resp lower epi-limit) (also called Γ-limits) is defined as the function $\mathscr{T} - ls_e f_n$ (resp $\mathscr{T} - li_e f_n$) whose epigraph is the lower limit (resp upper limit) of the sequence of epigraphs of the f_n's.*

The following formulas give analytic means for these limits (see [15] Definition 4.1):

$$\mathcal{T}-ls_e f_n(x) = \sup_{V\in\mathcal{N}(x)} \limsup_n \inf_{x'\in V} f_n(x'),$$
$$\mathcal{T}-li_e f_n(x) = \sup_{V\in\mathcal{N}(x)} \liminf_n \inf_{x'\in V} f_n(x').$$

In the sequel, $(x_n) \xrightarrow{\mathcal{T}} x$ means that the sequence $(x_n)_n$ $\mathcal{T}$-converges to x; we will use frequently the next sequential version of the definition of the $\mathcal{T}$-lower epi-limit (the two notions coincide when every point has a countable base of neighbourhoods [15]).

Definition 2.4 *Given a sequence $(f_n)_n$ of $\overline{\mathbb{R}}$-valued functions defined on $(\mathcal{X},\mathcal{T})$, and $\mathcal{Z} = \mathcal{X}^{\mathbb{N}}$, the sequential $\mathcal{T}$-lower epi-limit $f = seq\,\mathcal{T}-li_e f_n$ is defined by:*

$$f(x) = \inf_{\{(x_n)_n\in\mathcal{Z}:\ (x_n)\xrightarrow{\mathcal{T}} x\}} \liminf_n f_n(x_n).$$

The links between epi-limits and conjugacy have been studied in the convex case in [42], [37] and [41], [1], [55]; moreover in [48], [49], [50] there is links between epi-limits, variational convergences, usual operations and conjugacy. In order to prove the main results of this article (in section 6), we need to obtain few preliminar properties of the Fenchel-Moreau conjugate of the sequential weak-lower epi-limit of a sequence of nonconvex functions (Proposition 2.5, Theorem 2.9, Corollary 2.13), $\mathcal{X}_{\mathcal{T}}$ will denote the topological space $(\mathcal{X},\mathcal{T})$ and the symbol $\mathcal{W}_b^*(\mathcal{X}^*,\mathcal{X}_{\mathcal{T}})$ (in abbreviate form $\mathcal{W}_b^*$) denotes the topology on $\mathcal{X}^*$ of uniform convergence on the symmetric sequentially $\mathcal{T}$-compact sets of $\mathcal{X}$. By similarity with the case where $\mathcal{X}_{\mathcal{T}}$ is a Banach space endowed with its strong topology (see [32] §18 D, [52] Theorem 1.13) it is called *bounded weak star topology* or *bounded weak* topology*.

Proposition 2.5 *Let $(\mathcal{X},\mathcal{T})$ be a locally convex topological linear space. If $f = seq\,\mathcal{T}-li_e f_n$ then:*

$$f^* \le \mathcal{W}_b^* - ls_e f_n^*.$$

Proof of Proposition 2.5.

Lemma 2.6 *Let $\mathcal{K}$ be the set of all sequentially $\mathcal{T}$-compact sets. Setting for* $K \in \mathcal{K}$, $\iota_K(x) = 0$ *if $x \in K$, $+\infty$ if $x \notin K$, $f_n^K = f_n + \iota_K$, $f = seq\,\mathcal{T}-li_e f_n$, then:*

$$f = \inf_{K\in\mathcal{K}} seq\,\mathcal{T}-li_e f_n^K.$$

Proof of Lemma 2.6. Let $g = \inf_{K\in\mathcal{K}} seq\,\mathcal{T}-li_e f_n^K$. Since for every $K \in \mathcal{K}$, $f_n \le f_n^K$, we have $f \le seq\,\mathcal{T}-li_e f_n^K$, therefore $f \le g$. Conversely, suppose that $f(x) < r$. Then there exists a sequence $(x_n)_n$ $\mathcal{T}$-converging to x such that $\liminf_n f_n(x_n) < r$. Setting $K = \{x\} \cup \{x_n,\ n \in \mathbb{N}\} \in \mathcal{K}$, it is clear that $g(x) \le seq\,\mathcal{T}-li_e f_n^K(x) \le \liminf_n f_n(x_n) < r$, hence $g \le f$.

Lemma 2.7 *If $f = seq\,\mathcal{T}-li_e f_n$, then $f^*(x^*) \le \limsup_n f_n^*(x^*)$ for all $x^* \in \mathcal{X}^*$.*

Proof of Lemma 2.7. Let us suppose that $\limsup_n f_n^*(x^*) < r$. Then for n sufficiently large: $\langle x^*,.\rangle - r < f_n$ and as a consequence: $\langle x^*,.\rangle - r \le f$ or equivalently $f^* \le r$. □

Lemma 2.8 *Let $\mathscr{K}$ and f_n^{K} be as in Lemma 2.6. For every* $\mathrm{K} \in \mathscr{K}$*, one has:*

$$\limsup_n (f_n^{\mathrm{K}})^* \leq \mathscr{W}_b^* - ls_e f_n^*.$$

Proof of Lemma 2.8. Recall that given two extended real valued functions f, g defined on $\mathscr{X}$, the infimal convolution $f \Box g$ is (classically) defined by the formula:

$$(f \Box g)(x) = \inf_{y \in \mathscr{X}} f(x-y) + g(y),$$

one has $(f \Box g)^* = f^* + g^*$ and $(f+g)^* \leq f^* \Box g^*$. For each sequentially $\mathscr{T}$-compact set K, setting $\mathrm{L} = \mathrm{K} \cup -\mathrm{K}$, we get

$$(f_n^{\mathrm{K}})^* = (f_n + \iota_{\mathrm{K}})^* \leq f_n^* \Box \iota_{\mathrm{K}}{}^* \leq f_n^* \Box \iota_{\mathrm{L}}{}^* .$$

Moreover the family of sets $\mathrm{V}(\mathrm{L},\epsilon) = \{v^* : \iota_{\mathrm{L}}{}^*(v^*) \leq \epsilon\}$, $\mathrm{K} \in \mathscr{K}$, $\epsilon > 0$, is a base of $\mathscr{W}_b^*$-neighbourhoods of the origin. For every $v^* \in \mathrm{V}(\mathrm{L},\epsilon)$, we obtain:

$$(f_n^{\mathrm{K}})^*(x^*) \leq (f_n^* \Box \iota_{\mathrm{L}}{}^*)(x^*) \leq f_n^*(x^* + v^*) + \epsilon,$$

thus

$$(f_n^{\mathrm{K}})^*(x^*) \leq \inf_{v^* \in \mathrm{V}(\mathrm{L},\epsilon)} f_n^*(x^* + v^*) + \epsilon$$

and

$$\limsup_n (f_n^{\mathrm{K}})^*(x^*) \leq \limsup_n \inf_{v^* \in \mathrm{V}(\mathrm{L},\epsilon)} f_n^*(x^* + v^*) + \epsilon \leq \mathscr{W}_b^* - ls_e f_n^*(x^*) + \epsilon,$$

and since ϵ is arbitrary, the proof of Lemma 2.8 is complete. $\Box$

End of the proof of Proposition 2.5. From Lemma 2.6 $f = \inf_{\mathrm{K} \in \mathscr{K}} seq\,\mathscr{T} - li_e f_n^{\mathrm{K}}$, hence due to Lemmas 2.7, 2.8 we get:

$$f^* = \sup_{\mathrm{K} \in \mathscr{K}} (seq\,\mathscr{T} - li_e f_n^{\mathrm{K}})^* \leq \sup_{\mathrm{K} \in \mathscr{K}} \limsup_n (f_n^{\mathrm{K}})^* \leq \mathscr{W}_b^* - ls_e f_n^* . \ \Box$$

Theorem 2.9 *Let $(\mathscr{X}, \mathscr{T})$ be a locally convex topological linear space, $\mathscr{W}_b^*$ be the topology on $\mathscr{X}^*$ of uniform convergence on $\mathscr{T}$-compact sets of $\mathscr{X}$. Given a sequence $(f_n)_n$ of elements of $\overline{\mathbb{R}}^{\mathscr{X}}$ bounded below by a function $h \in \overline{\mathbb{R}}^{\mathscr{X}}$ inf-sequentially $\mathscr{T}$-compact for every slope, then setting $f = seq\,\mathscr{T} - li_e f_n$, we have:*

$$f^* = \limsup_n f_n^* = \mathscr{W}_b^* - ls_e f_n^* .$$

Corollary 2.10 *Under the assumptions of Theorem 2.9, for any topology τ stronger or equal than $\mathscr{W}_b^*$ we have: $\tau - ls_e f_n^* = f^* = \limsup_n f_n^*$.*

Proof of Corollary 2.10. From Theorem 2.9, we obtain the chain of inequalities

$$\limsup_n f_n^* = \mathscr{W}_b^* - ls_e f_n^* \leq \tau - ls_e f_n^* \leq \limsup_n f_n^* = f^* . \Box$$

Proof of Theorem 2.9. Let us first prove the following lemmas:

Lemma 2.11 *Let* K *be a sequentially $\mathcal{T}$-compact set. If* $(g_n)_n$ *is a sequence of* $\overline{\mathbb{R}}$*-valued functions defined on* K*, and* $g = seq\,\mathcal{T} - li_e g_n$*, then:*

$$\inf_{x\in K} g(x) \le \liminf_n \inf_{x\in K} g_n(x)$$

Proof of Lemma 2.11. Let $x_n \in K$ such $r_n = \inf_{x\in K} g_n(x) \ge g_n(x_n) - \frac{1}{n}$, and let $r > \liminf_n r_n$. Extracting subsequences we can find subsequences $(r_{n_k})_k$ such $\lim_k r_{n_k} < r$ and $(x_{n_k})_k$ $\mathcal{T}$-converging to $x \in K$. Then $g(x) \le \liminf_k g_{n_k}(x_{n_k}) \le \lim_k r_{n_k} + \frac{1}{n_k} < r$. □

Lemma 2.12 *Let* $h : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ *be a sequentially $\mathcal{T}$-inf-compact function. Given a sequence* $(f_n)_n$ *of elements of* $\overline{\mathbb{R}}^{\mathcal{X}}$ *bounded below by* h*, setting* $f = seq\,\mathcal{T} - li_e f_n$*, we have:*

$$f^*(0) = \limsup_n f_n^*(0) = \mathcal{W}_b^* - ls_e f_n^*(0).$$

Proof of Lemma 2.12. Let us first give the proof with the additional assumption:

$$\text{(H)} \quad \limsup_n \inf_{x\in\mathcal{X}} f_n(x) < \infty\,.$$

With (H) let $r \in \mathbb{R}$ and $(x_n)_n$ be a sequence such $(f_n(x_n))_n$ is bounded above by r. Since $f_n \ge h$ for every integer n, one has $\inf_{x\in\mathcal{X}} f_n(x) = \inf_{x\in h^{\le r}} f_n(x)$ and $h^{\le r}$ is nonempty sequentially $\mathcal{T}$-compact. Extracting from $(x_n)_n$ a $\mathcal{T}$-converging subsequence, we remark that $f^{\le r} \ne \emptyset$. Moreover since h is $\mathcal{T}$ sequentially lower semicontinuous, we have $f \ge h$ and also

$$\inf_{x\in\mathcal{X}} f(x) = \inf_{x\in f^{\le r}} f(x) = \inf_{x\in h^{\le r}} f(x)\,.$$

From Lemma 2.11 we obtain:

$$\inf_{x\in\mathcal{X}} f(x) = \inf_{x\in h^{\le r}} f(x) \le \liminf_n \inf_{x\in h^{\le r}} f_n(x) = \liminf_n \inf_{x\in\mathcal{X}} f_n(x).$$

Therefore:

$$f^*(0) = -\inf_{x\in\mathcal{X}} f(x) \ge -\liminf_n \inf_{x\in\mathcal{X}} f_n(x) = \limsup_n - \inf_{x\in\mathcal{X}} f_n(x) = \limsup_n f_n^*(0)\,,$$

hence, from Proposition 2.5:

$$f^*(0) \ge \limsup_n f_n^*(0) \ge \mathcal{W}_b^* - ls_e f_n^*(0) \ge f^*(0).$$

Proof of Lemma 2.12 without assumption (H). Let us consider a subsequence $(n_k)_k$ such that $\limsup_n f_n^*(0) = \lim_k f_{n_k}^*(0)$. If $\infty = \limsup_k \inf_{x\in\mathcal{X}} f_{n_k}(x) = -\lim_k f_{n_k}^*(0)$, using Proposition 2.5 the conclusion stems from the relations

$$f^*(0) \ge \lim_k f_{n_k}^*(0) = \limsup_n f_n^*(0) = -\infty = \mathcal{W}_b^* - ls_e f_n^*(0) \ge f^*(0)\,.$$

Suppose now that $\limsup_k \inf_{x\in\mathscr{X}} f_{n_k}(x) < \infty$. Define $g := seq\,\mathscr{T} - li_e f_{n_k} \geq f$. Since the sequence $(f_{n_k})_k$ satisfies the assumption (H), then from the first part of the proof and Proposition 2.5 we have:

$$f^*(0) \geq g^*(0) = \limsup_k f^*_{n_k}(0) = \limsup_n f^*_n(0) \geq \mathscr{W}^*_b - ls_e f^*_n(0) \geq f^*(0)\,.$$

This ends the proof of Lemma 2.12. □

End of the proof of Theorem 2.9. Given x^* in $\mathscr{X}^*$, set $g_n = f_n - \langle x^*, .\rangle$ and $l = h - \langle x^*, .\rangle$. Then l is a sequentially $\mathscr{T}$ inf-compact function, and for every n, $g_n \geq l$. Moreover since $g^*_n(y^*) = f^*_n(x^* + y^*)$, $g^*_n(0) = f^*_n(x^*)$, $\mathscr{W}^*_b - ls_e g^*_n(0) = \mathscr{W}^*_b - ls_e f^*_n(x^*)$ and $g = seq\,\mathscr{T} - li_e g_n = f - \langle x^*, .\rangle$. Applying Lemma 2.12 we obtain:

$$f^*(x^*) = g^*(0) = \limsup_n g^*_n(0) = \limsup_n f^*_n(x^*) = \mathscr{W}^*_b - ls_e g^*_n(0) = \mathscr{W}^*_b - ls_e f^*_n(x^*).$$

The proof of Theorem 2.9 is complete. □

The symbol σ denotes the weak topology $\sigma(\mathscr{X}, \mathscr{X}^*)$ on $\mathscr{X}$. When $\mathscr{X}$ is a Banach space, considering the case $\mathscr{T} = \sigma(,\mathscr{X}, \mathscr{X}^*)$, due to Eberlein-Smulian's theorem [8] IV §5 section 3* Theorem 2, [32] §18 Corollary A and Theorem B, the σ-compact sets are the sequentially σ-compact sets. Krein's result ([8] IV §5 section 5* Theorem 1, [32] §19 Theorem E) asserts that the closed convex hull of a weakly compact set is weakly compact too. Therefore the topology $\mathscr{W}^*_b$ on $\mathscr{X}^*$ coincide with the Mackey topology $\tau(\mathscr{X}^*, \mathscr{X})$ on $\mathscr{X}^*$ (of uniform convergence on weakly convex compact sets of $\mathscr{X}$) which is coarser than or equal to the topology of the dual norm $\|.\|_*$. An immediate consequence of Theorem 2.9 and Corollary 2.10 is:

Corollary 2.13 *Let $(\mathscr{X}, \|.\|)$ be a Banach space, τ^* be the Mackey topology $\tau(\mathscr{X}^*, \mathscr{X})$ on X^* and let $h : \mathscr{X} \to \mathbb{R} \cup \{\infty\}$ be an inf σ-compact function for every slope. Given a sequence $(f_n)_n$ of elements of $\overline{\mathbb{R}}^{\mathscr{X}^*}$ bounded below by h, setting $f = seq\,\sigma - li_e f_n$ then*

$$f^* = \limsup_n f^*_n = \tau^* - ls_e f^*_n = \|.\|_* - ls_e f^*_n\,.$$

The symbol σ^* denotes the weak star topology $\sigma(\mathscr{X}, \mathscr{X})$ on $\mathscr{X}^*$. The case of $(\mathscr{X}^*, \sigma^*)$ for the sequential lower epi-limit in convex case is treated in [2] Theorem 7.5.1 and the real lower epi-limit in convex case is considered in [49] Theorem 2.

Corollary 2.14 *Let $(\mathscr{X}, \|.\|)$ be a separable Banach space, and a sequence $(f_n)_n$ of elements of $\overline{\mathbb{R}}^{\mathscr{X}^*}$ bounded below by an inf σ^*-compact function for every slope, setting $f = seq\,\sigma^* - li_e f_n$ then for the duality between $\mathscr{X}^*$ and $\mathscr{X}$, with $f = seq\,\sigma^* - li_e f_n$,*

$$f^* = \limsup_n f^*_n = \|.\| - ls_e f^*_n\,.$$

Proof of Corollary 2.14. We take $\mathscr{T} = \sigma(\mathscr{X}^*, \mathscr{X})$ on X^*. When $\mathscr{X}$ is a separable Banach space it is well known that the σ^* compact sets are metrisable, thus the σ^* sequentially compact sets are the σ^* compact sets. Since the unit ball of $\mathscr{X}^*$ is a σ^* compact set, the associated topology $\mathscr{W}^*_b$ on $\mathscr{X}$ is the topology of uniform convergence on bounded sets of $\mathscr{X}^*$, that is the norm topology of $\mathscr{X}$. □

3 Biting Lemma, Biting convergence and tightness.

In the sequel $(\Omega, \mathbb{T}, \mu)$ is a measure space endowed with a σ-finite positive measure μ and with a tribe $\mathbb{T}$. For a measurable subset $A \in \mathbb{T}$, set $A^c = \{\omega \in \Omega, \omega \notin A\}$ and 1_A stands for the characteristic function of A: $1_A(\omega) = 1$ if $\omega \in A$, 0 if $\omega \notin A$. Given two measurable $\overline{\mathbb{R}}$-valued functions u and v denote by $\{u \geq v\}$ the set $\{\omega \in \Omega : u(\omega) \geq v(\omega)\}$. Given a topological space (E, τ) with Borel tribe $\mathscr{B}(E)$, the space $\mathscr{L}_0(E)$ is the space of τ-measurable E-valued functions. We will say that the sequence $(x_n)_n$ of elements of $\mathscr{L}_0(E)$ converges almost everywhere to x if there exists a negligible set N such for every $\omega \in N^c$, the sequence $(x_n(\omega))_n$ τ-converges to $x(\omega)$. Let $L_0(\Omega, E)$ be the space of classes of measurable functions (for μ-almost everywhere equality) defined on Ω and with values in E. It is customary to use the abuse of notation which consists to identify x and its class $[x]$, we will do it. When the topology τ is associated to a distance d the function defined on $L_0(\Omega, E)^2$ by $d_\mu(x, y) = \displaystyle\int_\Omega \frac{d(x(\omega), y(\omega))}{1 + d(x(\omega), y(\omega))} \alpha(\omega) d\mu(\omega)$ (where α is any positive valued integrable function) is a distance, and the topology associated is the topology of convergence in local μ-measure, that is convergence in measure on each set of finite μ-measure. Hereafter E is a separable Banach space. Let $L_p(\Omega, E, \mu)$, $1 \leq p \leq \infty$, be the Lebesgue-Bochner space of classes of p-μ integrable functions (μ-essentially bounded functions if $p = \infty$) defined on Ω with values in E and endowed with its strong natural topology. When there is no ambiguity with respect to the measure we denote it by $L_p(\Omega, E)$. $\|x\|_p$ is the usual norm of an element x of $L_p(\Omega, E)$, where for $1 \leq p < \infty$, $\|x\|_p^p = \displaystyle\int_\Omega \|x\|^p d\mu$. Given a map $v : \Omega \to \overline{\mathbb{R}}$, set $v^+ = \sup(0, v)$ and $v^- = (-v)^+$. The upper integral I_v or $\int_\Omega^* v d\mu$ of v is defined by:

$$I_v = \int_\Omega^* v d\mu = \inf\{\int_\Omega u d\mu, u \in L_1(\Omega, \mathbb{R}), u \geq v \ \ \mu - a.e\} = I_{v^+} - I_{v^-}$$

with the convention $+\infty - \infty = +\infty$. A function $\phi : E \to \overline{\mathbb{R}}_+$ is a Young function if it is convex even continuous at 0 with $\phi(0) = 0$ and verifies $\lim_{\|e\|_E \to \infty} \phi(e) = +\infty$. When $\lim_{\|e\|_E \to \infty} \frac{\phi(e)}{\|e\|} = +\infty$, ϕ is said strongly coercive. In the sequel we refer to a function $f : \Omega \times E \to \overline{\mathbb{R}}$ as an integrand. For every $\omega \in \Omega$ let $f_\omega = f(\omega, .)$. An integrand f is said to be *convex* respectively *even* if for every $\omega \in \Omega$ the function f_ω is convex (respectively even). The integrand f is said to be *measurable* if it is measurable when $\Omega \times E$ is endowed with the tribe $\mathbb{T} \otimes \mathscr{B}(E)$. Given two integrands f and g we write $f \leq g$ when there exists a negligible set N such that for every $(\omega, e) \in N^c \times E$, $f(\omega, e) \leq g(\omega, e)$.

A Young integrand ϕ is an integrand such that for every $\omega \in \Omega$, $\phi(\omega, .)$ is a Young function. If α is a positive valued integrable function, an integrand ϕ is said to be an α-Young integrand if there exists a non decreasing convex strongly coercive function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ verifying $\psi(0) = 0$ and: $\phi(\omega, e) = \alpha(\omega)\psi(\alpha^{-1}(\omega).\|e\|)$. Given a Young integrand $\phi : \Omega \times E \to \overline{\mathbb{R}}_+$, for every $t > 0$ we denote $\phi_t(\omega, e) = \phi(\omega, te)$. We will say that an integrand f is *of Nagumo type* if f is non negative, measurable and if for every $\omega \in \Omega$ f_ω *is inf-σ-compact for every slope.* When E is reflexive every α-Young integrand is a Nagumo integrand. Given $x \in L_0(\Omega, E)$, and an integrand f, we will denote by $f(x)$ the map $\omega \mapsto f(\omega, x(\omega))$. The integral functional I_f

associated to an integrand f is the functional defined at some point x of $L_0(\Omega, E)$ by:

$$I_f(x) = I_{f(x)} = \int_\Omega^* f(x)\, d\mu\,.$$

Let us recall the following definitions.

Definition 3.1 *([39]) Let* X *be a subset of* $L_p(\Omega, E)$ *with* $1 \le p < \infty$. X *is said to be* p*-equi-integrable (equi-integrable if* $p = 1$*), if for every positive number* ϵ*, there exist a positive constant* η *and a measurable set* K *of finite measure such that:*
(1) $\sup\limits_{x\in X} \|x1_A\|_p < \epsilon$ *for every measurable set* A *satisfying* $\mu(A) \le \eta$.
(2) $\sup\limits_{x\in X} \|x1_{K^c}\|_p < \epsilon$.

Some authors use only the first property as definition of equi-integrability see [21], Definition 2.23 for example. In order to measure the lack of equi-integrability let us introduce the following notion:

Definition 3.2 *(see [26]) Let* Σ *be the collection of all decreasing sequences* $\sigma = (S_k)_k$ *of measurable sets* S_k *with a negligible intersection. Given a sequence* $(u_n)_n$ *of measurable* $\overline{\mathbb{R}}$*-valued functions it is convenient to use the index of equi-integrability* $\delta^+_\mu((u_n)_n)$ *defined by:*

$$\delta^+_\mu((u_n)_n) = \sup_{\sigma\in\Sigma,\, \sigma=(S_k)_k} \limsup_k \sup_{n\ge k} \int_{S_k}^* u_n\, d\mu\,.$$

When there is no ambiguity on the measure we note $\delta^+((u_n)_n) := \delta^+_\mu((u_n)_n)$

Recall the following result about the index (of equi-integrability):

Theorem 3.3 *(see [26] Proposition 1.7 and Corollary 1.9)*
(a) *A sequence* $(u_n)_n$ *of integrable functions is equi-integrable in the sense of Definition 3.1 if and only if* $\delta^+((\|u_n\|)_n) = 0$.
(b) *Suppose the measure* μ *is atomless. A sequence* $(u_n)_n$ *of measurable functions is eventually uniformly integrable if and only if* $\delta^+((\|u_n\|)_n) = 0$.

Theorem and Definition 3.4 *De la Vallée Poussin's Theorem ([33] page 33, [19], [44] II-5,[16] Chapter 2, Theorems 22 and 25, see [58] Theorem 16.8). A subset* X *of* $L_1(\Omega, E)$ *is said uniformly integrable when one of the following equivalent conditions is satisfied:*
(a) X *is bounded and equi-integrable in* $L_1(\Omega, E)$,
(b) *There exists a positive valued integrable function* α *and a* α*-Young integrand* $\phi = \psi_\alpha$ *such that* $\sup_{x\in X} I_\phi(x) \le 1$.
(c) *For every integrable positive valued function* β, $\lim\limits_n \sup\limits_{x\in X} \int_{\{\|x\|\ge n\beta\}} \|x\|\, d\mu = 0\,.$
When the measure μ *is finite,(see [44] II-5), in the assertion* (b) *and* (c) *it suffices to consider the case* $\alpha = 1 = \beta$.

Proof. The equivalences $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ are well-known when the measure is finite and $\alpha = 1$. In the σ-finite case, for every positive valued integrable function α, first let us remark that

since the μ-negligible sets are the αμ-negligible sets, taking the assertion (a) has definition of uniform integrability, from Thorem 3.3 (a), we deduce that X is uniformly integrable in $L_1(\Omega, E, \mu)$ if and only if $\alpha^{-1}X$ is uniformly integrable in $L_1(\Omega, E, \alpha\mu)$. Therefore $(a) \Leftrightarrow (b)$. Moreover For every integrable positive valued function β, and for every integer n and $x \in X$,

$$\int_{\{\|x\| \geq n\beta\}} \|x\| d\mu = \int_{\{\|\beta^{-1}x\| \geq n\}} \|\beta^{-1}x\| \beta d\mu,$$

thus X verifies (c) in $L_1(\Omega, E, \mu)$ if and only if $\beta^{-1}X$ verifies (c) with "$\beta = 1$" in $L_1(\Omega, E, \beta\mu)$. Therefore $(a) \Leftrightarrow (c)$. □

Let us give now the statement of the Biting Lemma (valid in the σ-finite case):

Theorem 3.5 *([59] Lemma 3.4, [11] Theorem 6.1.4). Let $(x_n)_n$ be a bounded sequence in $L_1(\Omega, E)$. There exists a subsequence $(x_{n_k})_k$ and a decreasing sequence $(A_k)_k$ of measurable sets with a negligible intersection such that the sequence $(x_{n_k} 1_{A_k^c})_k$ is equi-integrable*

Proof of Theorem 3.5. The result is true when μ is finite valued ([59] Lemma 3.4, [11] Theorem 6.1.4). If μ is σ-finite there exists a positive valued integrable function α. The measure $\nu(A) = \int_A \alpha d\mu$ is finite valued. Let $(x_n)_n$ be a bounded sequence in $L_1(\Omega, E, \mu)$. The sequence $(\alpha^{-1}x_n)_n$ is a bounded sequence in $L_1(\Omega, E, \nu)$. There exists a subsequence $(\alpha^{-1}x_{n_k})_k$ and a decreasing sequence $(A_k)_k$ of measurable sets with a ν-negligible intersection such that the sequence $(\alpha^{-1}x_{n_k} 1_{A_k^c})_k$ is ν-equi-integrable. The ν-negligible sets being exactly the μ-negligible sets we deduce with Theorem 3.3 (a) that the sequence $(x_{n_k} 1_{A_k^c})_k$ is μ-equi-integrable. □

Definition 3.6 *Given a sequence $(x_n)_n$ of E-valued measurable functions defined on Ω, we will say that $(x_n)_n$ converges to a measurable E-valued function x in the Biting sense if there exists an increasing covering (up to a negligible set) $(\Omega_k)_k$ of Ω by measurable sets such that for all k the sequence $(x_n|\Omega_k)_n$ of restrictions to Ω_k $\sigma(L_1(\Omega_k, E, \mu), L_1(\Omega_k, E, \mu)^*)$-converges to $x|\Omega_k$.*

Corollary 3.7 *([21], [11] Remark 6.1.5), Suppose E is reflexive. Then any bounded sequence in $L_1(\Omega, E)$ admits a Biting converging subsequence to an integrable function.*

Proof: It is well-known ([17]) that in this case $L_1(\Omega, E)^* = L_\infty(\Omega, E^*)$, and in $L_1(\Omega, E)$ any bounded equi-integrable sequence admits a weakly converging subsequence. Applying the first part of Theorem 3.5 and extracting from $(x_{n_k} 1_{A_k^c})_k$ a weakly converging sequence $(x_{n_{k_l}} 1_{A_{k_l}^c})_l$ to an integrable function x, the sequence $(x_{n_{k_l}})_l$ converges in the Biting sense: the restrictions on each $A_{k_l}^c$ converge weakly to the restriction of the function x. And for every integer k

$$\|x 1_{\Lambda_k}\|_1 \leq \liminf_n \|x_n 1_{\Lambda_k}\|_1 \leq \liminf_n \|x_n\|_1 ,$$

Since the norm is weakly semicontinuous and the sequence is norm bounded:

$$\|x\|_1 \leq \liminf_k \|x 1_{\Lambda_k}\|_1 \leq \liminf_n \|x_n\|_1 < \infty . \ \square$$

Corollary 3.8 *Suppose* E *is reflexive. Let* $1 \leq p \leq \infty$ *and* β *be a measurable positive valued function. Every bounded sequence in* $\mathrm{L}_p(\Omega,\mathrm{E},\beta\mu)$ *admits a Biting converging subsequence with a Biting limit in* $\mathrm{L}_p(\Omega,\mathrm{E},\beta\mu)$).

Proof of Corollary 3.8. First let us give a proof when $\beta = 1$. When $p = 1$ it is exactly the Biting Lemma in σ-finite case above. When Ω is of finite measure since $\mathrm{L}_p(\Omega,\mathrm{E})$ is topologically included in $\mathrm{L}_1(\Omega,\mathrm{E})$, the result of the existence of a Biting-converging subsequence is an immediate consequence of the classical Biting Lemma. First let us show that the sequence admits a Biting convergent subsequence in σ-finite case. There exists a bounded integrable function α positive valued. Let a bounded sequence $(x_m)_m$ of p-integrable elements. Set $y_m = \alpha^{\frac{-1}{p}} x_m$, then the sequence $(y_m)_m$ is bounded in $\mathrm{L}_p(\Omega,\mathrm{E},\alpha\mu)$ thus in $\mathrm{L}_1(\Omega,\mathrm{E},\alpha\mu)$. Due to the Biting Lemma there exists $y \in \mathrm{L}_1(\Omega_k,\mathrm{E},\alpha\mu)$ an increasing covering $(\Omega_k)_k$ of Ω by measurable sets, a subsequence $(y_{m_n})_n$ such for all integer k the sequence of the restrictions of the y_{m_n} to Ω_k weakly converges in $\mathrm{L}_1(\Omega_k,\mathrm{E},\alpha\mu)$ to the restriction to Ω_k of y. That is for all $x^* \in \mathrm{L}_\infty(\Omega_k,\mathrm{E}^*)$,

$$\lim_n \int_{\Omega_k} \langle y_{m_n}, x^*\rangle \alpha d\mu = \int_{\Omega_k} \langle y, x^*\rangle \alpha d\mu$$

equivalently:

$$\lim_n \int_{\Omega_k} \langle x_{m_n}\alpha^{1-\frac{1}{p}}, x^*\rangle d\mu = \int_{\Omega_k} \langle y\alpha, x^*\rangle d\mu.$$

Setting $\Lambda_k = \{\omega \in \Omega_k : \alpha(\omega) \geq \frac{1}{k}\}$, it is clear that $(\Lambda_k)_k$ is an increasing covering of Ω and for every $x^* \in \mathrm{L}_\infty(\Lambda_k,\mathrm{E}^*)$, $y^* = \alpha^{\frac{1}{p}-1}.x^*$ is an element of $\mathrm{L}_\infty(\Lambda_k,\mathrm{E}^*)$, thus:

$$\lim_n \int_{\Lambda_k} \langle x_{m_n}, x^*\rangle d\mu = \lim_n \int_{\Lambda_k} \langle x_{m_n}, \alpha^{1-\frac{1}{p}} y^*\rangle d\mu = \int_{\Lambda_k} \langle y\alpha, y^*\rangle d\mu = \int_{\Lambda_k} \langle y\alpha^{\frac{1}{p}}, x^*\rangle d\mu\,.$$

This proves that the sequence $(x_{m_n})_n$ Biting converges to $x = y\alpha^{\frac{1}{p}}$. The assertion $x \in \mathrm{L}_p(\Omega,\mathrm{E})$ is a consequence of the Corollary 3.11 bellow. The proof of Corollary 3.8 is complete in the case $\beta = 1$.

Let β be a measurable positive valued function, the measure $\nu = \alpha\mu$ is σ-finite. If the sequence $(x_m)_m$ is bounded in $\mathrm{L}_p(\Omega,\mathrm{E},\nu)$ applying the first part of the proof we deduce the existence of a Biting-converging subsequence $(x_{m_n})_n$ to an element of $\mathrm{L}_p(\Omega,\mathrm{E},\nu)$. That is there exists an increasing covering $(\Omega_k)_k$ of Ω by measurable sets such that for all k the sequence $(x_n|\Omega_k)_n$ of restrictions to Ω_k $\sigma(\mathrm{L}_1(\Omega_k,\mathrm{E},\nu),\mathrm{L}_1(\Omega_k,\mathrm{E},\nu)^*)$-converges to $x|\Omega_k$. Define for all positive integer k, $\Lambda_k = \Omega_k \cap \{\omega : k^{-1} \leq \beta(\omega) \leq k\}$. Since β is positive valued, the ν-negligible sets are the μ-negligible sets. Moreover $(\Lambda_k)_k$ is an increasing covering of Ω by measurable sets such that for all k the sequence $(x_n|\Lambda_k)_n$ of restrictions to Λ_k $\sigma(\mathrm{L}_1(\Lambda_k,\mathrm{E},\mu),\mathrm{L}_1(\Lambda_k,\mathrm{E},\mu)^*)$-converges to $x|\Lambda_k$. □

Proposition 3.9 *Any convex* $\mathrm{C} \subset \mathrm{L}_0(\Omega,\mathrm{E})$ *closed for the convergence in local measure is Biting sequentially closed.*

The proof of Proposition 3.9 is an immediate consequence of the following Lemma:

Lemma 3.10 *If* $(x_m)_m$ *Biting-converges to* x *there exists a sequence* $(\overline{x}_m)_m$ *converging in local measure to* x *such that for all integer* m, $\overline{x}_m \in co\{x_n,\ n \geq m\}$

Proof of Lemma3.10. Given α a positive valued integrable function less than 1, the topology of convergence in local measure is defined by the distance:

$$d_\mu(x,y) = \int_\Omega \frac{\|x(\omega)-y(\omega)\|}{1+\|x(\omega)-y(\omega)\|}\alpha(\omega)d\mu(\omega)\ .$$

Since the sequence $(x_m)_m$ Biting converges to x there exists an increasing covering $(\Omega_k)_k$ of Ω such that for every integer k the sequence $(x_m 1_{\Omega_k})_m$ weakly converges in $L_1(\Omega,E)$ to $x1_{\Omega_k}$. Pick an integer k_m such that $\int_{\Omega^c_{k_m}} \alpha d\mu < \frac{1}{3m}$. Due to Mazur's Lemma, $x1_{\Omega_{k_m}}$ is in the strong closure of $co\{x_n 1_{\Omega_{k_m}},\ n \geq m\}$, thus there exists an element $\overline{x}_m \in co\{x_n,\ n \geq m\}$ such that $\|(x-\overline{x}_m)1_{\Omega_{k_m}}\|_1 < \frac{1}{3m}$. But

$$d_\mu(x,\overline{x}_m) \leq d_\mu(x, x1_{\Omega_{k_m}}) + d_\mu(x1_{\Omega_{k_m}}, \overline{x}_m 1_{\Omega_{k_m}}) + d_\mu(\overline{x}_m 1_{\Omega_{k_m}}, \overline{x}_m)$$

therefore:

$$d_\mu(x,\overline{x}_m) \leq 2\sup_z d_\mu(z1_{\Omega^c_{k_m}}, 0) + d_\mu(x1_{\Omega_{k_m}}, \overline{x}1_{\Omega_{k_m}})\ ,$$

and we get:

$$d_\mu(x,\overline{x}_m) \leq 2\int_{\Omega^c_{k_m}} \alpha d\mu + \int_{\Omega_{k_m}} \frac{\|x-\overline{x}_m\|}{1+\|x-\overline{x}_m\|}\alpha d\mu < 2\frac{1}{3m} + \|(x-\overline{x}_m)1_{\Omega_{k_m}}\|_1 < 3\frac{1}{3m} = \frac{1}{m}\ .$$

This proves that the sequence $(\overline{x}_m)_m$ converges in local measure to x. The proof of Lemma 3.10 is complete. $\square$

Since by Fatou's Lemma every norm closed ball of $L_p(\Omega,E)$ is closed for the convergence in local measure it follows:

Corollary 3.11 *Let* $1 \leq p \leq \infty$. *The norm closed balls of* $L_p(\Omega,E)$ *are sequentially Biting closed.*

Definition 3.12 *(see [5]) A subset* X *of* $L_0(\Omega,E)$ *is said to be Nagumo tight if there exists a Nagumo integrand h such* $\sup_{x\in X}\int_\Omega h(x)d\mu < \infty$.

Remark 3.13

If M is a E-valued multifunction with nonempty weakly compact values and with $\mathbb{T}\otimes\mathscr{B}(E)$-measurable graph, the integrand $f(\omega,e) = \iota_{M(\omega)}(e)$ is measurable, then f is an example of Nagumo integrand. As a consequence, for a such multifunction, the set of measurable almost everywhere selections of M is an example of Nagumo tight set.
Let us give another useful example of Nagumo tight sets.

Proposition 3.14 *If* E *is a reflexive Banach space, then every converging sequence in the Biting sense is Nagumo tight.*

Proof of the Proposition 3.14. When E is reflexive, every integrand of α-Nagumo type is a Nagumo integrand. By the Dunford-Pettis Theorem, every weakly compact set X in $L_1(\Omega,E)$ is uniformly integrable, thus from Theorem 3.4 (b), X is Nagumo tight. Therefore every

weakly converging sequence in $L_1(\Omega, E)$ is Nagumo tight. Let a sequence $(x_n)_n$ converging in the Biting sense. If $(\Omega_k)_k$ is the sequence of measurable sets appearing in Definition 3.6, let $\Lambda_0 = \Omega_0$ and for $k \geq 1$, $\Lambda_k = \Omega_k \backslash \Omega_{k-1}$. Since every weakly compact set X in $L_1(\Lambda_k, E)$ is uniformly integrable, for each integer k there exists a Nagumo integrand h_k defined on $\Lambda_k \times E$ such that: $\sup_n \int_{\Lambda_k} h_k(x_n)\,d\mu \leq 2^{-k-1}$. Define $\overline{h_k}(\omega, e) = h_k(\omega, e)$ if $\omega \in \Lambda_k$ and 0 if not; $h = \sum_k \overline{h_k}$, then h is a Nagumo integrand which satisfies

$$\sup_n \int_\Omega h(x_n)\,d\mu \leq \sum_k 2^{-k-1} = 1\,.$$

This proves that the sequence $(x_n)_n$ is Nagumo tight and ends the proof of the Proposition. □

Definition 3.15 *(see [11] Section 6 and Lemma 6.1.1) A subset* X *of* $L_0(\Omega, E)$ *is said to be weakly flexibly tight if for every set* A *of finite measure, for every* $\epsilon > 0$, *there exists a measurable multifunction (see Definition 4.1)* M_ϵ *with nonempty weakly compact values in* E *such that for all* $x \in X$,

$$\mu(A \cap \{\omega \in \Omega : x(\omega) \notin M_\epsilon(\omega)\}) \leq \epsilon\,.$$

The following result makes a link between Nagumo tightness and the notion of weak flexible tightness used in [11] Theorem 8.1.6.

Proposition 3.16 *Let* E *be a separable Banach space. A countable weakly flexibly tight subset of* $L_0(\Omega, E)$ *is Nagumo tight.*

Proof of Proposition 3.16. Suppose that $X = \{x_n, n \in \mathbb{N}\}$ is a countable weakly flexibly tight subset. First, let us prove the following lemma:

Lemma 3.17 *The following assertions hold:*
(a) *For every measurable set* A *of finite measure, for each* $\epsilon > 0$ *there exist a measurable multifunction* L_ϵ *with weakly compact values a measurable set* $A_\epsilon \subset A$ *with* $\mu(A_\epsilon) \leq \epsilon$ *and for all* $x \in X$, *for all* $\omega \in A_\epsilon^c$, $x(\omega) \in L_\epsilon(\omega)$.
(b) *Proposition 3.16 holds if* Ω *is of finite measure. More precisely there exists a measurable multifunction* M *with weakly compact values such that every* $x \in X$ *is an almost everywhere selection of* M.

Proof of Lemma 3.17. Let $\epsilon > 0$. For every integer n keep the multifunction $M_{\frac{\epsilon}{2^{n+1}}}$ of Definition 3.15. The measurable sets $A_n = \{\omega \in A : x_n(\omega) \notin M_{\frac{\epsilon}{2^{n+1}}}\}$ and $A_\epsilon = \bigcup_n A_n$ satisfies $\mu(A_n) \leq \frac{\epsilon}{2^{n+1}}$ and $\mu(A_\epsilon) \leq \epsilon$. If $L_\epsilon = \bigcap_n M_{\frac{\epsilon}{2^{n+1}}}$, then L_ϵ is measurable with weakly compact values. L_ϵ and A_ϵ satisfy the assertion (a) of Lemma 3.17.
Suppose Ω has finite measure. In order to prove the second assertion, let us build a multifunction M in the following way. Set $A_0 = \Omega$. From the first assertion, there exist a measurable set A_1 such $\mu(A_1) \leq \frac{\mu(\Omega)}{2}$ a measurable multifunction K_1 with weakly compact values such for every $\omega \in A_0 \backslash A_1$, for every $x \in X$, $x(\omega) \in K_1(\omega)$. Suppose build A_n and K_n such $\mu(A_n) \leq \frac{\mu(\Omega)}{2^n}$, K_n is measurable with weakly compact values such for every $\omega \in A_{n-1} \backslash A_n$, for every $x \in X$, $x(\omega) \in K_n(\omega)$. From assertion ($a$), applied to the measure space A_n, there exist a measurable set $A_{n+1} \subset A_n$ a measurable multifunction K_{n+1} with weakly compact values defined

on A_n such $\mu(A_{n+1}) \leq \frac{\mu(\Omega)}{2^{n+1}}$ and such for all $\omega \in A_n \backslash A_{n+1}$, for every $x \in X$, $x(\omega) \in K_{n+1}(\omega)$. Set $A_\infty = \bigcap_n A_n$ (then $\mu(A_\infty) = 0$), $M(\omega) = K_n(\omega)$ if $\omega \in A_{n-1} \backslash A_n$, $\{0\}$ if $\omega \in A_\infty$. Then M has nonempty weakly compact values, is measurable in sense of Definition 4.2, and by construction for every $\omega \in A^c_\infty$, for every $x \in X$, $x(\omega) \in M(\omega)$. Due to the Remark 3.13, the set X is Nagumo tight. □

End of the proof of Proposition 3.16. When the measure μ is σ-finite, let $(\Omega_p)_p$ be an increasing covering of Ω by measurable sets of finite measure. Using the above lemma, for each integer p there exists a measurable multifunction M_p with nonempty weakly compact values defined on Ω_p such for every $x \in X$, for almost every $\omega \in \Omega_p$, $x(\omega) \in M_p(\omega)$. Define then the multifunction M by $M(\omega) = M_0(\omega)$ if $\omega \in \Omega_0$, $M(\omega) = M_p(\omega)$ if $\omega \in \Omega_p \backslash \Omega_{p-1}$. By construction the multifunction M is measurable with weakly compact values. Moreover since X is countable, every element of X is an almost everywhere selection of M and Remark 3.13 allows to conclude. □

4 Measurability and polarity

In the sequel the tribe $\mathbb{T}$ is supposed to be μ-complete and E is a locally convex space. The epigraph multifunction of an integrand $f : \Omega \times E \to \overline{\mathbb{R}}$ is the multifunction $epi\ f$ defined by $epi\ f(\omega) = epi f_\omega$. Let E^* be the topological dual of E. The Fenchel-Moreau conjugate of f is the integrand f^* defined on $\Omega \times E^*$ by $f^*(\omega, e^*) = (f_\omega)^*(e^*)$. A Suslin space is a continuous image of a metrisable separable complete space. Let (E, τ) be a Suslin locally convex space often denoted by E_τ. Let $\sigma = \sigma(E, E^*)$ be the weak topology on E. On the topological dual E^*, let $\sigma^* = \sigma(E^*, E)$ be the weak star topology on E^*. Notice that if E is a separable Banach space, then E_σ and $E^*_{\sigma^*}$ are examples of Suslin locally convex spaces. The tribe $\mathscr{B}(E)$ is the Borel tribe of E_τ, $\mathscr{B}(E_\sigma)$ is the Borel tribe of E_σ and $\mathscr{B}(E^*_{\sigma^*})$ is the Borel tribe of $E^*_{\sigma^*}$. A Suslin space is hereditary Lindelöf (that is with the property that every covering of an open set by open sets admits a countable subcovering). Therefore if E, F are Suslin spaces then $\mathscr{B}(E \times F) = \mathscr{B}(E) \times \mathscr{B}(F)$. Recall the following notions.

Definition 4.1 *([53], [9] VII. 1) Let* F *be a Suslin space. A closed and* F*-valued multifunction* M *defined on* Ω *is said to be measurable if its graph is* $\mathbb{T} \otimes \mathscr{B}(F)$ *measurable.*

Definition 4.2 *([9] VII. 1) Let* (E, τ) *be a Suslin space and let* $f : \Omega \times E \to \overline{\mathbb{R}}$ *be an integrand. The integrand* f *is called normal on* $\Omega \times E_\tau$ *if it is* $\mathbb{T} \otimes \mathscr{B}(E_\tau)$ *measurable and for every* $\omega \in \Omega$ *the function* f_ω *is* τ*-lower semicontinuous.*

Proposition 4.3 *(see [53] Proposition 2 and [9] Corollary VII-2) Let* E *be a separable Banach space and let* $f : \Omega \times E \to \overline{\mathbb{R}}$ *be a* $\mathbb{T} \otimes \mathscr{B}(E)$ *measurable integrand. The Fenchel-Moreau conjugate integrand* f^* *is a convex normal integrand on* $\Omega \times E^*_{\sigma^*}$*, and the restriction of the biconjugate* f^{**} *to* $\Omega \times E$ *is a convex normal integrand on* $\Omega \times E_\sigma$.

Proof of Proposition 4.3. It is an immediate consequence of the following lemma.

Lemma 4.4 *(see [60] Lemma 8, [9] Corollary VII-2) Let* (E, τ) *be a Suslin locally convex space with topological dual* E^* *and let* $f : \Omega \times E \to \overline{\mathbb{R}}$ *be a* $\mathbb{T} \otimes \mathscr{B}(E_\tau)$*-measurable integrand. The Fenchel-Moreau conjugate integrand* f^* *is a convex normal integrand on* $\Omega \times E^*_{\sigma^*}$.

By the use of the projection Theorem and a Castaing representation of the epigraph multifunction of f, the proofs of [60] Lemma 8, [9] Corollary VII-2 are valid even when f is only supposed to be a $\mathbb{T}\otimes\mathscr{B}(\mathrm{E}_\tau)$-measurable integrand. When E is equipped with the norm topology it is a Suslin locally convex space. Moreover $\mathrm{E}^*_{\sigma^*}$ is a Suslin locally convex space space too with topological dual E. Therefore the two parts of Proposition 4.3 are easy consequences of Lemma 4.4. The proof of Proposition 4.3 is complete. □

Definition 4.5 *A sequence $(f_n)_n$ of extended real valued integrands is said to be quasi inf-σ-compact for every slope if for every $\omega \in \Omega$ the sequence of functions $(f_n(\omega,.))_n$ is eventually bounded below by a function that is inf-σ-compact for every slope.*

Theorem 4.6 *Let E be a separable Banach space and $(f_n)_n$ be a sequence of $\mathbb{T}\otimes\mathscr{B}(\mathrm{E})$ measurable integrands defined on $\Omega\times\mathrm{E}$. The pointwise weak lower sequential epi-limit integrand f is defined for all $(\omega,e)\in\Omega\times\mathrm{E}$ by $f(\omega,e) = seq\,\sigma - li_e f_{n_\omega}(e)$. Under one of the following assumptions the integrand f^* is normal on $\Omega\times\mathrm{E}^*_{\sigma^*}$ (and moreover f^{**} is normal on $\Omega\times\mathrm{E}_\sigma$):*
(a) The sequence $(f_n)_n$ is quasi inf-σ-compact for every slope in the sense of Definition 4.5.
(b) The Banach E is reflexive and separable.

Proof of Theorem 4.6. Due to Proposition 4.3 it suffices to proves that f^* is normal on $\Omega\times\mathrm{E}^*_{\sigma^*}$. We begin with the following lemmas:

Lemma 4.7 *Let $(\mathrm{E},\|.\|)$ be a separable Banach space and let $(f_n)_n$ be a sequence of $\mathbb{T}\otimes\mathscr{B}(\mathrm{E})$ measurable integrands defined on $\Omega\times\mathrm{E}$ which is quasi inf-σ-compact for every slope. If $f = seq\,\sigma - li_e f_n$, then f^* is normal on $\Omega\times\mathrm{E}^*_{\sigma^*}$.*

Proof of Lemma 4.7. Due to Proposition 4.3 the integrands f_n^* are normal integrands. From Corollary 2.13 and Definition 4.5 we have $\limsup_n f_n^* = f^*$. From Definition 4.2 f^* is a normal integrand on $\Omega\times\mathrm{E}^*_{\sigma^*}$. □

Lemma 4.8 *Let $(\mathrm{E},\|.\|)$ be a separable reflexive Banach space and let $(f_n)_n$ be a sequence of $\mathbb{T}\otimes\mathscr{B}(\mathrm{E})$ measurable integrands defined on $\Omega\times\mathrm{E}$ and bounded below by some measurable function $h:\Omega\to\mathbb{R}$. If $f = seq\,\sigma - li_e f_n$, then f^* is a normal integrand on $\Omega\times\mathrm{E}^*_{\sigma^*}$.*

Proof of Lemma 4.8. The non negative integrands $g_n = f_n - h$ are $\mathbb{T}\otimes\mathscr{B}(\mathrm{E})$ measurable. For each integer k define $g_{n,k} = g_n + k^{-1}\|.\|^2$ and $g_k = seq\,\sigma - li_e g_{n\,k}$. The integrands $g_{n,k}$ are $\mathbb{T}\otimes\mathscr{B}(\mathrm{E})$ measurable, bounded below by $k^{-1}\|.\|^2$, thus from Lemma 4.7, for each integer k, g_k^* is normal on $\Omega\times\mathrm{E}^*_{\sigma^*}$. Moreover

$$f - h = \inf_k seq\,\sigma - li_e g_{n,k} = \inf_k g_k\,,$$

therefore $f^* + h = \sup_k g_k^*$. The integrands g_k^* being normal, Definition 4.2 shows that the integrand $f^* = \sup_k g_k^* - h$ is a normal integrand on $\Omega\times\mathrm{E}^*_{\sigma^*}$. The proof of Lemma 4.8 is complete. □

End of the proof of Theorem 4.6. The first assertion is exactly Lemma 4.7. Suppose now

E is reflexive and separable. For each integer k define $f_{n,k} = \sup(f_n\ , -k)$. The $f_{n,k}$ are measurable and bounded below by k. If $f_k = seq\,\sigma - li_e f_{n,k}$ applying Lemma 4.8 we deduce that each ${f_k}^*$ is a normal integrand on $\Omega \times \mathrm{E}^*_{\sigma^*}$. But $f = \inf_k f_k$, then the integrand $f^* = \sup\limits_k {f_k}^*$ is a normal integrand on $\Omega \times \mathrm{E}^*_{\sigma^*}$. The proof of Theorem 4.6 is complete. □

Assume as precedently that the Banach E is separable. When E^* is equipped with the tribe $\mathscr{B}(\mathrm{E}^*_{\sigma^*})$, it is known that a function $x^* : \Omega \to \mathrm{E}^*$ is measurable if and only if it is scalarly measurable, that is for every $e \in \mathrm{E}$, the map $\omega \mapsto \langle x^*(\omega), e\rangle$ is $\mathbb{T}$-measurable. Indeed as a function $x^* : \Omega \to \mathrm{E}^*$ is scalarly measurable if and only if the inverse image of each weak* open translated half-space of E^* is $\mathbb{T}$-measurable, it suffices to prove that the tribe $\mathscr{B}'$ generated by the weak* open translated half-spaces coincide with $\mathscr{B}(\mathrm{E}^*_{\sigma^*})$. Since E is strongly separable the dual norm $\|x^*\|_*$ of any scalarly measurable E^*-valued function is $\mathbb{T}$-measurable. Let $\mathrm{L}_p(\Omega, \mathrm{E}^*_{\sigma^*})$ $1 \le p \le \infty$ be the set of equivalence classes for the equality almost everywhere of scalarly E^*-valued functions x^* such $\|x^*\|_* \in \mathrm{L}_p(\Omega, \mathbb{R})$. Then it is known (see [36] for $p = 1$, and [21] Theorem 2.112), that the strong dual of $\mathrm{L}_p(\Omega, \mathrm{E})$ is $\mathrm{L}_q(\Omega, \mathrm{E}^*_{\sigma^*})$ with the duality pairing

$$\langle x^*, x\rangle = \int_\Omega \langle x^*(\omega), x(\omega)\rangle\, d\mu(\omega)\ .$$

The following useful duality result is an immediate consequence of the main Theorem in [60] and [9] Theorem VII-7 (which is valid when the decomposable spaces considered are subspaces of scalarly measurable functions instead of subspaces of scalarly integrable functions).

Theorem 4.9 *(see [60] main Theorem, [9] Theorem VII-7) Let* E *be a separable Banach space and* $g : \Omega \times \mathrm{E}^* \to \overline{\mathbb{R}}$ *be a normal integrand on* $\Omega \times \mathrm{E}^*_{\sigma^*}$. *If* I_g *is finite at at least one point of* $\mathrm{L}_\infty(\Omega, \mathrm{E}^*_{\sigma^*})$ *then for the duality between* $\mathrm{L}_\infty(\Omega, \mathrm{E}^*_{\sigma^*})$ *and* $\mathrm{L}_1(\Omega, \mathrm{E})$, *for every* $x \in \mathrm{L}_1(\Omega, \mathrm{E})$ *we have* $\mathrm{I}^*_g(x) = \mathrm{I}_{g^*}(x)$.

5 Two properties of the Mackey topology $\tau(\mathrm{L}_\infty(\Omega, \mathrm{E}^*_{\sigma^*}), \mathrm{L}_1(\Omega, \mathrm{E}))$

In this section we suppose that the Banach E is separable. As above, $\mathscr{B}(\mathrm{E}^*_{\sigma^*})$ is the Borel tribe of $(\mathrm{E}^*, \sigma(\mathrm{E}^*, \mathrm{E}))$. Recall that the Mackey topology $\tau^* = \tau(\mathrm{L}_\infty(\Omega, \mathrm{E}^*_{\sigma^*}), \mathrm{L}_1(\Omega, \mathrm{E}))$ is the topology of uniform convergence on the $\sigma(\mathrm{L}_1(\Omega, \mathrm{E}), \mathrm{L}_\infty(\Omega, \mathrm{E}^*_{\sigma^*}))$-compacts convex sets of $\mathrm{L}_1(\Omega, \mathrm{E})$.

Proposition 5.1 *The* $\sigma(\mathrm{L}_1(\Omega, \mathrm{E}), \mathrm{L}_\infty(\Omega, \mathrm{E}^*_{\sigma^*}))$*-compact sets are uniformly integrable.*

Proof of Proposition 5.1. Since $\mathrm{L}_1(\Omega, \mathrm{E})^* = \mathrm{L}_\infty(\Omega, \mathrm{E}^*_{\sigma^*})$, due to Eberlein-Smulian Theorem, each $\sigma(\mathrm{L}_1(\Omega, \mathrm{E}), \mathrm{L}_\infty(\Omega, \mathrm{E}^*_{\sigma^*}))$-compact set is a $\sigma(\mathrm{L}_1(\Omega, \mathrm{E}), \mathrm{L}_\infty(\Omega, \mathrm{E}^*_{\sigma^*}))$-sequentially compact set therefore it is a $\sigma(\mathrm{L}_1(\Omega, \mathrm{E}), \mathrm{L}_\infty(\Omega, \mathrm{E}^*_{\sigma^*}))$-conditionally compact set (every sequence has a weakly Cauchy subsequence), and we use [17] IV Theorem 4. Other proof: see also [18]. □

Corollary 5.2 *If* E *is a separable Banach space, for every bounded sequence* (x^*_n) *in* $\mathrm{L}_\infty(\Omega, \mathrm{E}^*_{\sigma^*})$, *for every decreasing sequence* $(\mathrm{A}_n)_n$ *of measurable sets with a negligible intersection, the sequence* $(x^*_n 1_{\mathrm{A}_n})_n$ τ^**-converges to the origin.*

Proof of Corollary 5.2. The topology τ^* being the topology of uniform convergence on the convex $\sigma(L_1(\Omega,E),L_\infty(\Omega,E^*_{\sigma^*}))$-compact sets, Proposition 5.1 gives the result. Indeed, considering a $\sigma(L_1(\Omega,E),L_\infty(\Omega,E^*_{\sigma^*}))$-compact set K, it suffices to use Theorem 3.3 (*a*) and to remark that:

$$\sup_{x\in K}|\int_{A_n}\langle x,x_n^*\rangle d\mu|\le(\sup_n\|x_n^*\|_\infty)\sup_{x\in K}\|x1_{A_n}\|_1\,.\ \square$$

Proposition 5.3 *Let τ^* be the Mackey topology $\tau(L_\infty(\Omega,E^*_{\sigma^*}),L_1(\Omega,E))$. Given a sequence $(f_n)_n$ of extended real-valued $\mathbb{T}\otimes\mathscr{B}_{\sigma^*}(E^*)$ measurable integrands defined on $\Omega\times E^*$ such that there exist an element $x_0^*\in L_\infty(\Omega,E^*_{\sigma^*})$, an element $u_0\in L_1(\Omega,\mathbb{R})$ verifying for every integer n $f_n(x_0^*)\le u_0$, then with $f=\|.\|_*-ls_e f_n$ one has*

$$\tau^*-ls_e I_{f_n}\le I_f.$$

Proof of Proposition 5.3. Clearly considering $g_n(\omega,e)=f_n(\omega,x_0^*+e)-u_0(\omega)$, one can suppose that $x_0^*=0$ and $u_0=0$. Let us endows E^* with the dual norm $\|.\|_*$ and $E^*\times\mathbb{R}$ with the product norm and the tribe $\mathscr{B}(E^*_{\sigma^*}\times\mathbb{R})$. We denote the unit ball of E^* by B^* and the unit ball of $L_\infty(\Omega,E^*_{\sigma^*})$ by B_∞. Let $x^*\in L_\infty(\Omega,E^*_{\sigma^*})$, $r\in\mathbb{R}$ verifying $I_f(x^*)<r<\infty$. Then there exists $u\in epif$, integrable such $f(x^*)\le u$ and $\int ud\mu<r$. Due to Definition 2.3 for every $\omega\in\Omega$, $epif_\omega=\liminf_n epif_{n_\omega}$, hence the function $s_n(\omega)=d((x^*(\omega),u(\omega)),epif_{n_\omega})$ converges simply to 0.

Lemma 5.4 *The integrand $g(\omega,e^*,r)=\|x^*(\omega)-e^*\|_*+|u(\omega)-r|$ is $\mathbb{T}\otimes\mathscr{B}(E^*_{\sigma^*}\times\mathbb{R})$ measurable.*

Proof of Lemma 5.4. Let x^* be a scalarly measurable function. Since the map $(\omega,e^*)\mapsto e^*$ is $\mathbb{T}\otimes\mathscr{B}(E^*_{\sigma^*})$-measurable and the map $(\omega,e^*)\mapsto x^*(\omega)$ is measurable too, the map $(\omega,e^*)\mapsto x^*(\omega)-e^*$ is $\mathbb{T}\otimes\mathscr{B}(E^*_{\sigma^*})$-measurable. The dual norm $\|.\|_*$ being $\sigma(E^*,E)$-lower semicontinuous, the map $(\omega,e^*)\mapsto\|x^*(\omega)-e^*\|_*$ is $\mathbb{T}\otimes\mathscr{B}(E^*_{\sigma^*})$-measurable and the integrand

$$h:(\omega,e^*,r)\mapsto\|x^*(\omega)-e^*\|_*$$

is $\mathbb{T}\otimes\mathscr{B}(E^*_{\sigma^*}\times\mathbb{R})$-measurable. The Caratheodory integrand $(\omega,r)\mapsto|u(\omega)-r|$ being $\mathbb{T}\otimes\mathscr{B}(\mathbb{R})$-measurable, the integrand

$$k:(\omega,e^*,r)\mapsto|u(\omega)-r|$$

is $\mathbb{T}\otimes\mathscr{B}(E^*_{\sigma^*}\times\mathbb{R})$-measurable. Then $g=h+k$ is the sum of $\mathbb{T}\otimes\mathscr{B}(E^*_{\sigma^*}\times\mathbb{R})$ measurable integrands hence it is $\mathbb{T}\otimes\mathscr{B}(E^*_{\sigma^*}\times\mathbb{R})$ measurable. The proof of Lemma 5.4 is complete.

Since f_n is $\mathbb{T}\otimes\mathscr{B}(E^*_{\sigma^*})$-measurable, the multifunction $epif_n$ has a $\mathbb{T}\otimes\mathscr{B}(E^*_{\sigma^*}\times\mathbb{R})$-measurable graph and the above lemma ensures that g is $\mathbb{T}\otimes\mathscr{B}(E^*_{\sigma^*}\times\mathbb{R})$-measurable. Therefore, due to [9] Lemma III 39, the function s_n is $\mathbb{T}$-measurable. Let α be a function in $L_\infty(\Omega,\mathbb{R})\cap L_1(\Omega,\mathbb{R})$ with positive values such $\int\alpha d\mu=1$ and let $\epsilon=r-\int ud\mu$. Since every $\sigma(L_1(\Omega,E),L_\infty(\Omega,E^*_{\sigma^*}))$-compact set is bounded, the strong topology of $L_\infty(\Omega,E^*_{\sigma^*})$ is stronger or equal that τ^*. Let V and W be symmetric τ^*-neighbourhoods of 0 such $W+W\subset V$, and $0<\beta\le\frac{\epsilon}{2}$ such $2\beta\alpha B_\infty\subset W$.

Define $\Omega_n=\cap_{k\ge n}\{s_k<\beta\alpha\}$. Then $(\Omega_n)_n$ is an increasing covering of Ω by measurable sets. Due to Corollary 5.2, the sequence $(x^*1_{\Omega_n^c})_n$ τ^*-converges to 0. Thus there exists an integer n_W such $n\ge n_W\Rightarrow x^*1_{\Omega_n^c}\in W$. Since g is $\mathbb{T}\times\mathscr{B}(E^*_{\sigma^*}\times\mathbb{R})$-measurable, the multifunction

$$\Gamma_n(\omega)=\{(\omega,e^*,r):g(\omega,e^*,r)<s_n(\omega)+\frac{1}{n}\beta\alpha(\omega)\}$$

has a $\mathbb{T} \times \mathscr{B}(\mathrm{E}^*_{\sigma^*} \times \mathbb{R})$ measurable graph. On each Ω_n, $\Gamma_n \cap epi f_n$ has a $\mathbb{T} \times \mathscr{B}(\mathrm{E}^*_{\sigma^*} \times \mathbb{R})$ measurable graph and nonempty values. Due to [9] Theorem III 22, on each Ω_n, there exists a scalarly-measurable selection (x^*_n, u_n) of $epi f_n$ satisfying

$$\|x^* - x^*_n\|_* + |u - u_n| \leq 2\beta\alpha\,. \tag{2}$$

Define $\overline{x^*_n}(\omega) = x^*_n(\omega)$ if $\omega \in \Omega_n$, 0 otherwise; and $\overline{u_n}(\omega) = u_n(\omega)$ if $\omega \in \Omega_n$, 0 otherwise. Then due to (2):

$$n \geq n_{\mathrm{W}} \Rightarrow x^* - \overline{x^*_n} = (x^* - \overline{x^*_n})1_{\Omega_n} + x^* 1_{\Omega_n^c} \in 2\beta\alpha \mathrm{B}_\infty + \mathrm{W} \subset \mathrm{W} + \mathrm{W} \subset \mathrm{V}. \tag{3}$$

Since $f_n(0) \leq 0$ we get $f_n(\overline{x_n}) \leq \overline{u_n}$ and using (2):

$$\mathrm{I}_{f_n}(\overline{x^*_n}) \leq \int_{\Omega_n} u_n d\mu \leq \int_{\Omega_n} (u + 2\beta\alpha) d\mu \leq \int_{\Omega_n} (u + \epsilon\alpha) d\mu.$$

Hence with (3)

$$\limsup_n \inf_{y^* \in x^* + \mathrm{V}} \mathrm{I}_{f_n}(y^*) \leq \int_\Omega (u + \epsilon\alpha) d\mu = r\,,$$

and since V is arbitrary:

$$\tau^* - ls_e \mathrm{I}_{f_n}(x^*) \leq r.$$

Therefore for every x^*,

$$\tau^* - ls_e \mathrm{I}_{f_n}(x^*) \leq \mathrm{I}_f(x^*). \;\square$$

6 Convergence results

In this section E is a separable Banach space and we consider now a topological subspace $(\mathscr{X}, \mathscr{T})$ of $\mathrm{L}_0(\Omega, \mathrm{E})$, satisfying at $x \in \mathscr{X}$, as the usual topologies on the spaces L_p, $1 \leq p \leq \infty$, the following property:

(P) *Every sequence $\mathscr{T}$-converging to $x \in \mathscr{X}$ converges to x in the Biting sense.*

Definition 6.1 *Let $(\mathrm{I}, \leq)$ be a totally ordered set. We will say that a family $(f_i)_{i \in \mathrm{I}}$ of functions satisfies eventually a property ($\mathscr{P}$) if there exists $i_0 \in \mathrm{I}$ such that the family $(f_i)_{i \geq i_0}$ satisfies this property.*

Definition 6.2 *Given a sequence $(f_n)_n$ of $\overline{\mathbb{R}}$-valued integrands defined on $\Omega \times \mathrm{E}$, we will say that a sequence $(x_n)_n$ satisfies the boundedness property (with respect to $(f_n)_n$) if the sequence $(f^-_n(x_n))_n$ is eventually bounded in $\mathrm{L}_1(\Omega, \mathbb{R})$. The sequence $(x_n)_n$ is said to satisfy the lower compactness property (with respect to $(f_n)_n$) if the sequence $(f^-_n(x_n))_n$ is eventually uniformly integrable in $\mathrm{L}_1(\Omega, \mathbb{R})$.*

Theorem 6.3 *Let E be a separable Banach space and let $(f_n)_n$ be a sequence of $\mathbb{T} \otimes \mathscr{B}(\mathrm{E})$ measurable and $\overline{\mathbb{R}}$-valued integrands. Suppose that property (P) holds. Given $x \in \mathscr{X}$, a Nagumo tight sequence $(x_n)_n$ $\mathscr{T}$-converging to x and satisfying the boundedness property, then for $f = seq\,\sigma - li_e f_n$ one has:*

$$\liminf_n \mathrm{I}_{f_n}(x_n) \geq \mathrm{I}_{f^{**}}(x) - \delta^+((-f_n(x_n))_n)\,.$$

Corollary 6.4 *Let* E *be a separable Banach space and let* $(f_n)_n$ *be a sequence of* $\mathbb{T}\otimes\mathscr{B}(\mathrm{E})$ *measurable and* $\overline{\mathbb{R}}$*-valued integrands. Suppose that property* (P) *holds. Given* $x\in\mathscr{X}$*, a Nagumo tight sequence* $(x_n)_n$ $\mathscr{T}$*-converging to* x *and satisfying the lower compactness property, then with* $f = seq\,\sigma - li_e f_n$ *one has*

$$\liminf_n \mathrm{I}_{f_n}(x_n) \geq \mathrm{I}_{f^{**}}(x).$$

When E is is reflexive and separable, the assumptions on tightness are satisfied (Proposition 3.14) and we obtain:

Corollary 6.5 *Suppose* E *is reflexive and separable,* $x\in\mathscr{X}$*, and property* (P) *holds. Given a sequence* $(f_n)_n$ *of* $\mathbb{T}\otimes\mathscr{B}(\mathrm{E})$ *measurable integrands, then for every sequence* $(x_n)_n$ $\mathscr{T}$*-converging to* $x\in\mathscr{X}$ *and satisfying the boundedness property, the following inequality holds with* $f = seq\,\sigma - li_e f_n$,

$$\liminf_n \mathrm{I}_{f_n}(x_n) \geq \mathrm{I}_{f^{**}}(x) - \delta^+((-f_n(x_n))).$$

If in addition $(x_n)_n$ *satisfies the lower compactness property then,*

$$\liminf_n \mathrm{I}_{f_n}(x_n) \geq \mathrm{I}_{f^{**}}(x).$$

The following definition, when the sequence $(f_n)_n$ is constant, is exactly the criterion put in light by A. D. Ioffe in the study of the strong weak-semicontinuity [34].

Definition 6.6 *Given a sequence* $(f_n)_n$ *of* $\overline{\mathbb{R}}$*-valued integrands defined on* $\Omega\times\mathrm{E}$*, we will say that it satisfies the* $\mathscr{T}$*-Ioffe's criterion at* $x\in\mathscr{X}$ *if for every subsequence* $(f_{n_k})_k$ *and every sequence* $(x_k)_k$ $\mathscr{T}$*-converging to* x *such the sequence* $(\mathrm{I}_{f_{n_k}}(x_k))_k$ *is bounded above, the sequence* $(x_k)_k$ *has the lower compactness property with respect to* $(f_{n_k})_k$.

Corollary 6.7 *Suppose* E *is reflexive and separable,* $x\in\mathscr{X}$*, and property* (P) *holds. Given a sequence* $(f_n)_n$ *of measurable integrands satisfying the* $\mathscr{T}$*-Ioffe's criterion at* $x\in\mathscr{X}$*, then:*

$$seq\,\mathscr{T} - li_e \mathrm{I}_{f_n}(x) \geq \mathrm{I}_{f^{**}}(x).$$

There exists a converse of Corollary 6.7. Recall that a subset X of $\mathrm{L}_0(\Omega,\mathrm{E})$ is said to be decomposable, see [31], if given elements x, y of X, then for every measurable set A, $z = x1_{\mathrm{A}} + y1_{\mathrm{A}^c}$ is also an element of X. We will suppose that the topological space $(\mathscr{X},\mathscr{T})$ satisfies the following property:

(Q) *Given a sequence* $(\mathrm{A}_n)_n$ *of measurable subsets such for every measurable set* A *of finite measure* $\lim_n \mu(\mathrm{A}\cap\mathrm{A}_n) = 0$*, and two sequences* $(x_n)_n$ *and* $(y_n)_n$ $\mathscr{T}$*-converging to* $x\in\mathscr{X}$*, then the sequence* $(z_n)_n$*, with* $z_n = x_n 1_{\mathrm{A}_n} + y_n 1_{\mathrm{A}_n^c}$*, is* $\mathscr{T}$*-converging to* x.

Theorem 6.8 *(see [26] Theorem 4.2) Suppose the measure* μ *is atomless, the topological space* $(\mathscr{X},\mathscr{T})$ *is decomposable and satisfies property* (Q) *at some point* $x\in\mathscr{X}$*. Given a sequence* $(f_n)_n$ *of* $\overline{\mathbb{R}}$*-valued measurable integrands defined on* $\Omega\times\mathrm{E}$ *and a* $\mathbb{T}\otimes\mathscr{B}(\mathrm{E})$*-measurable integrand* $\overline{\mathbb{R}}$*-valued* f *satisfying:*
for every $\epsilon>0$ *there exists a sequence* $(y_n)_n$ $\mathscr{T}$*-converging to* x *such* $\delta^+((f_n^-(y_n))_n) \leq \epsilon$ *and* $\limsup_n \mathrm{I}_{f_n}(y_n) \leq \mathrm{I}_f(x) + \epsilon$.
Then if $-\infty < \mathrm{I}_f(x)$*, the Ioffe's criterion 6.6 is necessary to get the inequality*

$$\mathrm{I}_f(x) \leq seq\,\mathscr{T} - li_e \mathrm{I}_{f_n}(x)\,.$$

Notice the following simple statements of the Ioffe's criterion:

Lemma 6.9 *(see [26] Lemma 4.1) Suppose that there exists a $\mathscr{T}$-converging sequence $(y_n)_n$ to $x \in \mathrm{X}$ such the sequence $(\mathrm{I}_{f_n}(y_n))_n$ is bounded above.*
(a) the Ioffe's criterion is equivalent to the following:
"for every sequence $(x_n)_n$ $\mathscr{T}$-converging to x such that the sequence $(\mathrm{I}_{f_n}(x_n))_n$ is bounded above, the sequence $(x_n)_n$ has the lower compactness property with respect to $(f_n)_n$".
(b) Moreover if the measure is atomless and if the topological space $\mathscr{X}$ satisfies the property (Q), then the Ioffe's criterion is equivalent to
" every sequence $(x_n)_n$ $\mathscr{T}$-converging to x has the lower compactness property with respect to $(f_n)_n$".

Proof of Theorem 6.3. We begin with a sequence of preliminary propositions.

Proposition 6.10 *Given a sequence $(f_n)_n$ of measurable integrands bounded below by a Nagumo integrand, we have on $\mathscr{X} = \mathrm{L}_1(\Omega, \mathrm{E})$, when $\mathscr{T}$ is the weak topology and $f = seq\,\sigma - li_e f_n$:*

$$seq\,\mathscr{T} - li_e \mathrm{I}_{f_n} \geq \mathrm{I}_{f^{**}}.$$

Proof of Proposition 6.10. Let $g = \|.\|_* - ls_e f_n^*$ be the integrand pointwise upper epi-limit of the f_n^* for the topology of the dual norm $\|.\|_*$ defined by $g(\omega, e) = \|.\|_* - ls_e {f_n^*}_\omega(e)$. The sequence $(f_n)_n$ being bounded below by a Nagumo integrand and the topology of the dual norm on E^* being stronger than or equal to the Mackey topology $\tau(\mathrm{E}^*, \mathrm{E})$, then from Corollary 2.13, we have for (almost) every ω, $g_\omega = \limsup_n f_{n_\omega}^* = f_\omega^*$ therefore $g = f^*$ and from Theorem 4.6, g is a normal integrand on $\Omega \times \mathrm{E}^*_{\sigma^*}$. Since the f_n's are non negative, then for every integer n, $f_n^*(0) \leq 0$. From Proposition 5.3 with the Mackey topology $\tau^* = \tau(\mathrm{L}_\infty(\Omega, \mathrm{E}^*_{\sigma^*}), \mathrm{L}_1(\Omega, \mathrm{E}))$ (in case E is a Banach space with strongly separable dual, from [27] Corollary 4.8),

$$\tau^* - ls_e \mathrm{I}_{f_n^*} \leq \mathrm{I}_g\,.$$

Thus with Proposition 2.5,

$$(seq\,\sigma - li_e \mathrm{I}_{f_n})^* \leq \tau^* - ls_e \mathrm{I}_{f_n^*} \leq \mathrm{I}_g\,.$$

As a consequence:

$$(seq\,\sigma - li_e \mathrm{I}_{f_n})^{**} \geq (\mathrm{I}_g)^*\,.$$

and since $g(0) = \limsup f_n{}^*(0) \leq 0$, with [9] Theorem VII-7 (see Theorem 4.9) we get on $\mathrm{L}_1(\Omega, \mathrm{E})$:

$$(\mathrm{I}_g)^* = \mathrm{I}_{g^*}$$

and therefore:

$$(seq\,\sigma - li_e \mathrm{I}_{f_n})^{**} \geq \mathrm{I}_{g^*}.$$

But $g^* = f^{**}$, and as a consequence:

$$seq\,\sigma - li_e \mathrm{I}_{f_n} \geq (seq\,\sigma - li_e \mathrm{I}_{f_n})^{**} \geq \mathrm{I}_{f^{**}}.$$

This ends the proof of Proposition 6.10. □

Proposition 6.11 *Suppose $(f_n)_n$ is a sequence of non negative measurable integrands. A Nagumo tight sequence $(x_n)_n$ of elements of $\mathscr{X} = \mathrm{L}_1(\Omega, \mathrm{E})$ which weakly-converges to x satisfies*

$$\liminf_n \mathrm{I}_{f_n}(x_n) \geq \mathrm{I}_{f^{**}}(x).$$

Proof of Proposition 6.11. There exists a Nagumo integrand h such: $\sup_n \mathrm{I}_h(x_n) \leq 1$. Let us consider now for every $\epsilon > 0$ the integrand $g_n^\epsilon = f_n + \epsilon h$. Clearly, f_n being non negative, each g_n^ϵ is bounded below by ϵh. Therefore from Proposition 6.10, since we have with $g^\epsilon = seq\,\sigma - li_e g_n^\epsilon \geq seq\,\sigma - li_e f_n = f$, we get

$$\liminf_n \mathrm{I}_{g_n^\epsilon}(x_n) \geq seq\,\sigma - li_e \mathrm{I}_{g_n^\epsilon}(x) \geq \mathrm{I}_{g^{\epsilon**}}(x) \geq \mathrm{I}_{f^{**}}(x).$$

But since $\mathrm{I}_{f_n}(x_n) \geq \mathrm{I}_{g_n^\epsilon}(x_n) - \epsilon$, we deduce:

$$\liminf_n \mathrm{I}_{f_n}(x_n) \geq \liminf_n \mathrm{I}_{g_n^\epsilon}(x) - \epsilon \geq \mathrm{I}_{f^{**}}(x) - \epsilon.$$

The above inequality being valid for every $x \in \mathrm{L}_1(\Omega, \mathrm{E})$, for every tight sequence $(x_n)_n$ weakly converging to x, and for every $\epsilon > 0$, the proof of Proposition 6.11 is complete. $\square$

Proposition 6.12 *For any measurable set* K *of finite measure the conclusion of Corollary 6.4 holds when $\mathscr{X} = \mathrm{L}_1(\mathrm{K}, \mathrm{E})$ and $\mathscr{T}$ is the weak topology of $\mathscr{X}$.*

Proof of Proposition 6.12. Let a Nagumo tight sequence $(x_n)_n$ weakly-converging to $x \in \mathrm{L}_1(\mathrm{K}, \mathrm{E})$ and satisfying the lower compactness property and

$$\liminf_n \mathrm{I}_{f_n}(x_n) < r < \infty.$$

Due to the lower compactness property of $(x_n)_n$, the sequence $(f_n^-(x_n))_{n \geq m}$ is, for m large enough, uniformly integrable.

Setting $\mathrm{A}_{n,m} = \{f_n^-(x_n) \geq m\}$, and $r_m = \sup_{n \geq m} \int_{\mathrm{A}_{n,m}} f_n^-(x_n)\, d\mu$, we have $\lim_{m \to \infty} r_m = 0$.

For every integer m, define $f_n^m = \sup(f_n, -m)$, then

$$f_n(x_n) = f_n(x_n) 1_{\mathrm{A}_{n,m}^c} + f_n(x_n) 1_{\mathrm{A}_{n,m}} = f_n^m(x_n) 1_{\mathrm{A}_{n,m}^c} + f_n(x_n) 1_{\mathrm{A}_{n,m}}.$$

Since $f_n^m(x_n) 1_{\mathrm{A}_{n,m}} = -m 1_{\mathrm{A}_{n,m}}$, $f_n^m(x_n) 1_{\mathrm{A}_{n,m}^c} \geq f_n^m(x_n) 1_{\mathrm{A}_{n,m}^c} + f_n^m(x_n) 1_{\mathrm{A}_{n,m}} = f_n^m(x_n)$, one has

$$f_n(x_n) \geq f_n^m(x_n) - f_n^-(x_n) 1_{\mathrm{A}_{n,m}}.$$

Therefore for $n \geq m$,

$$\mathrm{I}_{f_n}(x_n) \geq \mathrm{I}_{f_n^m}(x_n) - r_m.$$

Using the above inequality and Proposition 6.11 with $f^m = seq\,\sigma - li_e f_n^m$ we deduce

$$r > \liminf_n \mathrm{I}_{f_n}(x_n) \geq \liminf_n \mathrm{I}_{f_n^m}(x_n) - r_m \geq \mathrm{I}_{f^{m**}}(x) - r_m \geq \mathrm{I}_{f^{**}}(x) - r_m,$$

and since $\lim_{m \to \infty} r_m = 0$, we obtain the desired inequality: $\mathrm{I}_{f^{**}}(x) \leq r$. The proof of Proposition 6.12 is complete. $\square$

End of the proof of Theorem 6.3.
Let a Nagumo tight sequence $(x_n)_n$ $\mathscr{T}$-converging to $x \in \mathscr{X}$ and satisfying the boundedness property. Suppose $\liminf_n \mathrm{I}_{f_n}(x_n) < r < \infty$. We can extract a subsequence $(x_{n_k})_k$ such

$$\limsup_k \mathrm{I}_{f_{n_k}}(x_{n_k}) < r < \infty\,.$$

From (P) the sequence $(x_{n_k})_k$ is Biting converging to x. The sequence $(f^-_{n_k}(x_{n_k}))_k$ being eventually bounded in $\mathrm{L}_1(\Omega,\mathbb{R})$, using the other form of the Biting Lemma, Corollary 3.7 in the σ-finite case with $\mathrm{E}=\mathbb{R}$, we can suppose (up to an extraction of a subsequence) that the sequences $(f^-_{n_k}(x_{n_k}))_k$ and $(x_{n_k})_k$ are converging in the Biting sense. Let $(\Omega_p)_p$ be a common increasing covering of Ω by measurable sets appearing in Definition 3.6. Since μ is σ-finite, we may suppose that the Ω_p's are of finite measure. By the Dunford-Pettis Theorem [21] Theorem 2.54 (see also [17], [19]) the sequence of restrictions $(f^-_{n_k}(x_{n_k})|\Omega_p)_k$ is uniformly integrable in the sense of Definition 3.4 in $\mathrm{L}_1(\Omega_p,\mathrm{E})$. This proves that for each integer p the sequence of restrictions $(x_{n_k}|\Omega_p)_k$ has the lower compactness property with respect to the sequence $(f_{n_k}|\Omega_p)_k$ of the restrictions to Ω_p of the integrands f_{n_k}.
The sequence $(f^-_n(x_n))_n$ being eventually bounded in $\mathrm{L}_1(\Omega,\mathbb{R})$, we deduce that the sequence $(\mathrm{I}_{f^+_{n_k}}(x_{n_k}))_k$ is eventually bounded above by some positive real number s. Clearly $f \le g = seq\,\sigma - li_e f^+_{n_k}$ and from Proposition 6.11 applied to the sequence $(f^+_{n_k})_k$, for every integer p on the space $\mathrm{L}_1(\Omega_p,\mathrm{E})$,

$$\int^*_{\Omega_p} f^{**}(x)^+ d\mu \le \int^*_{\Omega_p} g^{**}(x)\, d\mu \le \liminf_k \mathrm{I}_{f^+_{n_k}}(x_{n_k}) \le s < \infty\,.$$

The Monotone Convergence Theorem for the upper integral gives:

$$\int^*_{\Omega} f^{**}(x)^+ d\mu \le s < \infty\,.$$

For each integer p we have for $k \ge p$

$$\mathrm{I}_{f_{n_k}}(x_{n_k}) = \int^*_{\Omega_p} f_{n_k}(x_{n_k})\, d\mu + \int^*_{\Omega_p^c} f_{n_k}(x_{n_k})\, d\mu\,,$$

thus

$$\mathrm{I}_{f_{n_k}}(x_{n_k}) \ge \int^*_{\Omega_p} f_{n_k}(x_{n_k})\, d\mu + \inf_{k\ge p}\int^*_{\Omega_p^c} f_{n_k}(x_{n_k})\, d\mu\,,$$

hence

$$\limsup_k \mathrm{I}_{f_{n_k}}(x_{n_k}) \ge \liminf_k \int^*_{\Omega_p} f_{n_k}(x_{n_k})\, d\mu + \inf_{k\ge p}\int^*_{\Omega_p^c} f_{n_k}(x_{n_k})\, d\mu\,.$$

Therefore the use of Proposition 6.12 gives

$$r > \limsup_k \mathrm{I}_{f_{n_k}}(x_{n_k}) \ge \int^*_{\Omega_p} f^{**}(x)\, d\mu + \inf_{k\ge p}\int^*_{\Omega_p^c} f_{n_k}(x_{n_k})\, d\mu\,,$$

and we obtain

$$r > \mathrm{I}_{f^{**}}(x) - \int_{\Omega_p^c}^{*} f^{**}(x)^+ d\mu + \inf_{k\geq p} \int_{\Omega_p^c}^{*} f_{n_k}(x_{n_k}) d\mu \,.$$

Since $\mathrm{I}_{f^{**}(x)^+} < \infty$ we have $\lim_p \int_{\Omega_p^c}^{*} f^{**}(x)^+ d\mu = 0$, and we deduce:

$$r \geq \mathrm{I}_{f^{**}}(x) + \liminf_p \inf_{k\geq p} \int_{\Omega_p^c}^{*} f_{n_k}(x_{n_k}) d\mu \,.$$

Therefore:

$$r \geq \mathrm{I}_{f^{**}}(x) + \inf_{\sigma\in\Sigma,\, \sigma=(\mathrm{S}_p)_p} \liminf_p \inf_{k\geq p} \int_{\mathrm{S}_p}^{*} f_{n_k}(x_{n_k}) d\mu \,,$$

or

$$r \geq \mathrm{I}_{f^{**}}(x) - \sup_{\sigma\in\Sigma,\, \sigma=(\mathrm{S}_p)_p} \limsup_p \sup_{k\geq p} \int_{\mathrm{S}_p}^{*} -f_{n_k}(x_{n_k}) d\mu \,,$$

equivalently:

$$r \geq \mathrm{I}_{f^{**}}(x) - \delta^+((-f_{n_k}(x_{n_k}))_k).$$

Therefore, observing that if $(v_n)_n$ is a subsequence of $(u_n)_n$ we have $\delta^+((v_n)_n) \leq \delta^+((u_n)_n)$, we get

$$r \geq \mathrm{I}_{f^{**}}(x) - \delta^+((-f_n(x_n))_n).$$

The proof of Theorem 6.3 is complete. □

Corollary 6.4 is an immediate consequence of Theorem 6.3 and Theorem 3.3. Indeed since $-f_n(x_n) \leq f_n^-(x_n)$ we have $0 \leq \delta^+((-f_n(x_n))_n) \leq \delta^+((f_n^-(x_n))_n) = 0$.
The proof of Corollary 6.5 is an immediate consequence of Theorem 6.3, Corollary 6.4 and Proposition 3.14. □
Proof of Corollary 6.7.
Let us suppose now that the sequence $(f_n)_n$ satisfies the $\mathcal{T}$-Ioffe's criterion and:

$$r > seq\,\mathcal{T} - li_e \mathrm{I}_{f_n}(x) \,.$$

Let a subsequence $(f_{n_k})_k$ and a sequence $(x_k)_k$ $\mathcal{T}$-converging to x such $r > \sup_k \mathrm{I}_{f_{n_k}}(x_k)$. Since $(f_n)_n$ satisfies the $\mathcal{T}$-Ioffe's criterion at x, the sequence $(x_k)_k$ has the lower compactness property with respect to $(f_{n_k})_k$. If $g = seq\,\sigma - li_e f_{n_k}$ the last part of the Corollary 6.4 applied to the sequence $(f_{n_k})_k$ gives:

$$r > \liminf_k \mathrm{I}_{f_{n_k}}(x_k) \geq \mathrm{I}_{g^{**}}(x) \geq \mathrm{I}_{f^{**}}(x) \,.$$

This last inequality being valid for any $r > seq\,\mathcal{T} - li_e \mathrm{I}_{f_n}(x)$, the proof of Corollary 6.7 is complete. □

The following result is of interest even if $dim(\mathrm{E}) = 1$, in the sequel we will use it. It can be proved directly using the properties of uniform integrability and of convergence in local measure.

Lemma 6.13 *Suppose the Banach* E *is separable. Let a sequence* $(u_n)_n$ *of real valued measurable functions converging in local measure to* 0 *and an uniformly integrable sequence* $(z_n)_n$ *of* E*-valued measurable functions such that* $z_n = u_n.y_n$*. If the sequence* $(y_n)_n$ *is bounded in some* $L_p(\Omega, E, \beta\mu)$ *for some measurable positive valued function* β *and* $1 \le p \le \infty$*, then the sequence* $(z_n)_n$ *strongly converges to the origin of* $L_1(\Omega, E)$.

Alternative proof of Lemma 6.13. It suffices to prove that every subsequence $(z_{n_k})_k$ of $(z_n)_n$ admits a strongly converging subsequence to the origin. The sequence $(\|y_n\|)_n$ is bounded in some $L_p(\Omega, \mathbb{R}, \beta\mu)$ $1 \le p \le \infty$, due to Corollary 3.8, it is possible from any subsequence $(\|y_{n_k}\|)_k$ to extract a subsequence $(\|y_{n_{k_l}}\|)_l$ which converges in the Biting sense to some $\xi \in L_p(\Omega, \mathbb{R}, \beta\mu)$. Moreover by extraction of a subsequence one may suppose that $(u_{n_{k_l}})_l$ converges almost everywhere to 0. Let $z^* \in L_\infty(\Omega, \mathbb{R}, \mu)$, and let the integrands defined on $\Omega \times \mathbb{R}$ by

$$f_l(\omega, r) = u_{n_{k_l}}.r.z^*(\omega) \text{ and } f = 0 = li_e f_l \text{ , then } f_l(\|y_{n_{k_l}})\| = u_{n_{k_l}}\|y_{n_{k_l}}\|.z^* = \|z_{n_{k_l}}\|.z^* \text{ .}$$

Moreover for every measurable set A , $\int_A |f_l(\|y_{n_{k_l}}\|)| d\mu \le \|z^*\|_\infty.\|z_{n_{k_l}} 1_A\|_1$. The sequence $(z)_n$ being uniformly integrable, the sequence $(f_l(\|y_{n_{k_l}}\|))_l$ is uniformly integrable. Therefore the sequence $(\|y_{n_{k_l}}\|)_l$ has the lower compactness property with respect to $(f_l)_l$. Due to Corollary 6.5,

$$\liminf_l \int_\Omega \|z_{n_{k_l}}\|.z^* \, d\mu = \liminf_l I_{f_l}(\|y_{n_{k_l}}\|) \ge I_f(\xi) = 0.$$

This inequality being valid for every $z^* \in L_\infty(\Omega, \mathbb{R})$ we deduce that $(\|z_{n_{k_l}}\|)_n$ weakly (therefore strongly) converges to 0. The proof of Lemma 6.13 is complete. □

7 Sequential strong-weak lower semicontinuity.

In this section we consider a topological space (E, τ), a separable Banach space F, with norm $\|.\|_F$, and its weak topology σ_F. We will use two topological spaces $(\mathscr{X}, \mathscr{S})$ and $(\mathscr{Y}, \mathscr{T})$ such that $\mathscr{X} \subseteq L_0(\Omega, E)$ and $\mathscr{Y} \subseteq L_0(\Omega, F)$. The topologies $\mathscr{S}$ and $\mathscr{T}$ verify the following assumptions:

(H_1) Every sequence $\mathscr{S}$-converging to $x \in \mathscr{X}$ admits a subsequence which converges to x almost everywhere.

$(H_1^{'})$ Every sequence $\mathscr{S}$-converging to $x \in \mathscr{X}$ converges to x in local measure.

(H_2) Every sequence $\mathscr{T}$-converging to $y \in \mathscr{Y}$ is a converging sequence to y in the Biting sense (see Definition 3.6).

(H_3) The sets $\mathscr{X}$ and $\mathscr{Y}$ are decomposable. Moreover given a sequence $(A_n)_n$ of measurable subsets such for every measurable set A of finite measure $\lim_n \mu(A \cap A_n) = 0$, and a sequence $(x_n, y_n)_n$ of elements of $\mathscr{X} \times \mathscr{Y}$, $\mathscr{S} \times \mathscr{T}$-converging to $(x, y) \in \mathscr{X} \times \mathscr{Y}$, then the sequence $(w_n)_n$, with $w_n = (x, y)1_{A_n^c} + (x_n, y_n)1_{A_n}$ $\mathscr{S} \times \mathscr{T}$-converges to (x, y).

Given an integrand $f : \Omega \times \mathrm{E} \times \mathrm{F} \to \overline{\mathbb{R}}$, the integrand $f^{**} : \Omega \times \mathrm{E} \times \mathrm{F} \to \overline{\mathbb{R}}$ is in this section, the partial Fenchel-Moreau biconjugate defined for every $(\omega, e, e^{'}) \in \Omega \times \mathrm{E} \times \mathrm{F}$ by:

$$f_\omega^{**}(e,\, e^{'}) = f_\omega(e,\, .)^{**}(e^{'})\,.$$

Theorem 7.1 *Suppose that assumptions* (H_1) *and* (H_2) *hold. Let* $x \in \mathscr{X}$ *and* $y \in \mathscr{Y}$, *and let* $f : \Omega \times \mathrm{E} \times \mathrm{F} \to \overline{\mathbb{R}}$ *be a* $\mathbb{T} \otimes \mathscr{B}(\mathrm{E} \times \mathrm{F})$*-measurable integrand. Given a sequence* $((x_n, y_n))_n$ $\mathscr{S} \times \mathscr{T}$*-converging to* (x, y) *and satisfying:*
(a) *the sequence* $(y_n)_n$ *is Nagumo tight,*
(b) *the sequence* $(f^-(x_n, y_n))_n$ *is eventually bounded in* $\mathrm{L}_1(\Omega, \mathbb{R})$,
then with the $\tau \times \sigma_\mathrm{F}$ *sequential lower semicontinuous regularization* g *of* f *defined for every* $\omega \in \Omega$ *by* $g_\omega = seq\, \tau \times \sigma_\mathrm{F} - li_e f_\omega$,

$$\liminf_n \mathrm{I}_f(x_n, y_n) \geq \mathrm{I}_{g^{**}}(x, y) - \delta^+((-f(x_n, y_n))_n)\,. \tag{4}$$

One has in addition $\liminf_n \mathrm{I}_f(x_n, y_n) \geq \mathrm{I}_f(x, y)$, *when the additional conditions are fulfilled:*
(c) *the sequence* $((x_n, y_n))_n$ *satisfies the lower compactness property with respect to* f,
(d) *for almost every* $\omega \in \Omega$, $g_\omega^{**}(x(\omega), y(\omega)) = f_\omega(x(\omega),\, y(\omega))$.

Remark 7.2 *Remark (since in this case* $f = g$*), that the condition* (d) *is satisfied at* (x, y) *when the following two conditions hold for every* $\omega \in \Omega$,
(i) *the function* f_ω *is* $\tau \times \sigma_\mathrm{F}$*-sequentially lower semicontinuous at each point of* $\{x(\omega)\} \times \mathrm{F}$.
(j) $f_\omega^{**}(x(\omega),\, y(\omega)) = f_\omega(x(\omega),\, y(\omega))$.
Moreover the condition (d) *is satisfied at any* (x, z) *for* $z \in \mathrm{Y}$ *when* (i) *holds and we replace the condition* (j) *by:*
(k) $f_\omega(x(\omega), .)$ *is proper convex.*

(Indeed in this last case $f_\omega(x(\omega), .)$ is proper convex norm lower semicontinuous, hence we have $f_\omega(x(\omega), .) = f_\omega(x(\omega), .)^{**}$, [52] Theorem 3.44). Conditions (i) and (k) first appear in Balder's work on seminormality and sequential semicontinuity [5] Theorem 4.9 and Theorem 4.12. The authors in [11] chapter 8.1 Theorem 8.1.6 obtain a semicontinuity result with a mild assumption of type (i), conditions (k) and (c), a Nagumo tightness assumption too (see Proposition 3.16) and the topology $\mathscr{T}$ considered being $\sigma(\mathrm{L}_1(\Omega, \mathrm{F}), \mathrm{L}_\infty(\Omega, \mathrm{F}^*))$, moreover the proof is very distinct.

Remark 7.3 *In the condition* (i) *one cannot replace the weak topology of* F *by the strong topology as show the example 8.1.8 in [11].*

It may happen that condition (d) holds for f and fails for f^{**}: because the semicontinuity condition (i) fails as shows the following example.

Example 7.4 $\Omega = (0, 1)$ *endowed with the Lebesgue tribe and the Lebesgue measure* dt, *with* $\mathrm{E} = \mathrm{F} = \mathbb{R}$ *and the continuous integrand* f *is defined on* $(0, 1) \times \mathbb{R}^2$ *by*

$$f(\omega, s, t) = \max(-|s|.|t|\,, -1)\,.$$

Then the lower semicontinuous regularization g of f is equal to f, hence f verifies (i) at any point (x, y). Moreover when $s = 0$, $f(s,.)^{**} = f(s,.) = 0$: f verifies (j) (therefore (d)) at any point $(0, t) \in \{0\} \times \mathbb{R}$; but $f(s,.)^{**} = -1$ when $s \neq 0$. Therefore f^{**} is not lower semicontinuous at any point $(0, t) \in \{0\} \times \mathbb{R}$. The lower semicontinuous regularization h of f^{**} is $h(s, t) = -1 = h^{**}(s, t)$, therefore for any $(0, t) \in \{0\} \times \mathbb{R}$, $f^{**}(0, t) \neq h^{**}(0, t)$, this proves that the condition (d) fails for f^{**} at any point $(0, t) \in \{0\} \times \mathbb{R}$.

Proof of Theorem 7.1. Let $r \in \mathbb{R}$ such that $\liminf_n \mathrm{I}_{f_n}(x_n, y_n) < r$. Since (H_1) holds, extracting subsequences we may reduce to the case when the sequence $(x_n)_n$ converges almost everywhere to x and $\sup_n \mathrm{I}_{f_n}(x_n, y_n) < r$. Setting $f_n(\omega, e) = f(\omega, x_n(\omega), e)$, the sequence $(y_n)_n$ is supposed Nagumo tight and with (b) is supposed to verify the boundedness property with respect to the sequence of measurable integrands $(f_n)_n$. Due to assumption (H_2), the topological space $(\mathscr{Y}, \mathscr{T})$ verifies the property (P) of the previous section. The use of Theorem 6.3 gives, with $h = seq\, \sigma_{\mathrm{F}} - li_e f_n$:

$$r > \liminf_n \mathrm{I}_{f_n}(y_n) \geq \mathrm{I}_{h^{**}}(y) - \delta^+((-f(x_n, y_n))_n)\,.$$

The above inequality is valid for every $r > \liminf_n \mathrm{I}_f(x_n, y_n)$. Moreover we remark that in addition, $g(x,.) \leq h$, hence we have almost everywhere $g^{**}(x, y) \leq h^{**}(y)$. This proves the validity of the inequality (4).
If assumption (c) holds, then $0 \leq \delta^+((-f(x_n, y_n))_n) \leq \delta^+((f^-(x_n, y_n))_n) = 0$. The Assumption (d) and formula (4) gives the second semicontinuity result. The proof of Theorem 7.1 is complete. □

Definition 7.5 *(see [34]) A measurable integrand* $f : \Omega \times \mathrm{E} \times \mathrm{F} \to \overline{\mathbb{R}}$ *satisfies the Ioffe's criterion at* $(x, y) \in \mathscr{X} \times \mathscr{Y}$ *if for every sequence* $((x_n, y_n))_n$ $\mathscr{S} \times \mathscr{T}$*-converging to* (x, y) *such that the sequence* $(\mathrm{I}_f(x_n, y_n))_n$ *is bounded above, the sequence* $(f^-(x_n, y_n))_n$ *is uniformly integrable.*

The corollary below is an extension of the Ioffe's result [34].

Corollary 7.6 *Suppose* (E, τ) *is a metrisable topological space,* F *is a reflexive separable Banach space and assumptions* $(\mathrm{H}_1^{'})$ *and* (H_2) *hold. Let* $f : \Omega \times \mathrm{E} \times \mathrm{F} \to \overline{\mathbb{R}}$ *be a* $\mathbb{T} \otimes \mathscr{B}(\mathrm{E} \times \mathrm{F})$*-measurable integrand satisfying for almost every* $\omega \in \Omega$ *the conditions:*
(i) *the function* f_ω *is sequentially* $\tau \times \sigma_{\mathrm{F}}$*-lower semicontinuous at each point of* $\{x(\omega)\} \times \mathrm{F}$.
(j) $f_\omega^{**}(x(\omega),\ y(\omega)) = f_\omega(x(\omega),\ y(\omega))$.
Then, whenever the integral functional I_f *satisfies the Ioffe's criterion at* $(x, y) \in \mathscr{X} \times \mathscr{Y}$ *it is* $\mathscr{S} \times \mathscr{T}$*-sequentially lower semicontinuous at this point.*
Conversely, suppose that the measure μ *is atomless, assumption* (H_3) *holds and* $\mathrm{I}_f(x, y) \neq -\infty$. *Then the Ioffe's criterion at* (x, y) *is necessary for the sequential lower semicontinuity of* I_f *at this point.*

Proof of Corollary 7.6. Let us prove first the sufficiency part. When the conditions (i) and (j) hold, then due to the Remark 7.2 the condition (d) holds. Let $r \in \mathbb{R}$ and a sequence $((x_n, y_n))_n$ $\mathscr{S} \times \mathscr{T}$-converging to (x, y) such for every integer n

$$\mathrm{I}_f(x_n, y_n) \leq r\,.$$

The Ioffe criterion ensures that the sequence $((x_n, y_n))_n$ verifies the lower compactness property with respect to the integrand f. Due to (H_2) and Proposition 3.14 the sequence $(y_n)_n$ is Nagumo tight. Since $(\mathrm{H}_1^{'})$ holds and every sequence converging in local measure admits a subsequence converging almost everywhere ([39]) then (H_1) holds. Therefore from Theorem 7.1 we get:

$$\mathrm{I}_f(x,y) \le \liminf_n \mathrm{I}_{f_n}(x_n, y_n) \le r$$

This proves that the integral functional I_f is sequentially $\mathscr{S} \times \mathscr{T}$-lower semicontinuous at (x,y).

The proof of the necessity part is very similar to the proof given by A. D. Ioffe in [34] Theorem 1. Indeed if I_f is $\mathscr{S}\times\mathscr{T}$-sequentially lower semicontinuous at (x,y) and the Ioffe's criterion is not true, there exist a real number r, a sequence $((x_n, y_n))_n$ $\mathscr{S}\times\mathscr{T}$-converging to (x,y) such the sequence $(f^-((x_n,y_n)))_n$ is not uniformly integrable and for every n, $\mathrm{I}_f(x_n,y_n) \le r$. Since I_f is sequentially lower semicontinuous at (x,y) we get: $\mathrm{I}_f(x,y) \le r < \infty$, and by assumptions $\mathrm{I}_f(x,y) > -\infty$, thus $f(x,y)$ is integrable. The sequence $(f^-(x_n,y_n))_n$ being not uniformly integrable and the measure considered being atomless, with the help of Theorem 3.3 (b) ([26], Proposition 1.7), we have:

$$\delta^+((f^-(x_n,y_n))_n) > 0\,.$$

Therefore there exist $\epsilon > 0$, a subsequence $((x_{n_k}, y_{n_k}))_k$ and a decreasing sequence $(\mathrm{A}_k)_k$ with a negligible intersection satisfying:

$$\int_{\mathrm{A}_k}^{*} f^-(x_{n_k}, y_{n_k})\,d\mu \ge \epsilon\,.$$

Set $\mathrm{B}_k = \{f(x_{n_k}, y_{n_k}) \le 0\}$, $\mathrm{C}_k = \mathrm{A}_k \cap \mathrm{B}_k$, $(x_k^{'}, y_k^{'}) = (x,y)1_{\mathrm{C}_k^c} + (x_{n_k}, y_{n_k})1_{\mathrm{C}_k}$. Due to H_3, since $\mathrm{C}_k \subset \mathrm{A}_k$, the sequence $((x_k^{'}, y_k^{'}))_k$ is a sequence of elements of $\mathscr{X}\times\mathscr{Y}$ which $\mathscr{S}\times\mathscr{T}$-converges to (x,y) and satisfies eventually:

$$\mathrm{I}_f(x_k^{'}, y_k^{'}) - \mathrm{I}_f(x,y) = \int_{\mathrm{C}_k}^{*} (f(x_{n_k}, y_{n_k}) - f(x,y))\,d\mu = -\int_{\mathrm{A}_k}^{*} f^-(x_{n_k}, y_{n_k})\,d\mu - \int_{\mathrm{C}_k} f(x,y) \le -\frac{\epsilon}{2}\,.$$

This contradicts the sequential $\mathscr{S}\times\mathscr{T}$-lower semicontinuity of I_f at (x,y). The proof of Corollary 7.6 is complete. □

Remark 7.7 *Under the assumption* (Q) *on the topology* $\mathscr{S}\times\mathscr{T}$ *instead of* (H_3)*, the necessity part is also a consequence of Theorem 6.8.*

Remark 7.8 *It may happen that at a given point* (x,y) *the integral functional* I_f *is sequentially* $\mathscr{S}\times\mathscr{T}$*-lower semicontinuous at this point and the sequential* $\mathscr{S}\times\mathscr{T}$*-lower semicontinuity of* $\mathrm{I}_{f^{**}}$ *fails at the same point* (x,y)*.*

Indeed consider the integrand f of Example 7.4 and a point $(0,y) \in \{0\}\times \mathrm{L}_2((0,1),\mathbb{R})$ and keep $\mathscr{X} = \mathscr{Y} = \mathrm{L}_2((0,1),\mathbb{R})$. The integrand f verifies the condition (d) at $(0,y)$, let us prove that the integral functional I_f is sequentially strong-weak continuous on $\mathrm{L}_2((0,1),\mathbb{R}^2)$ at $(0,y)$. Let us give a direct proof. Given a strong-weak converging sequence $(x_n,y_n)_n$ to $(0,y)$, from

the Hölder inequality the sequence $(|x_n|.|y_n|)_n$ is uniformly integrable since $(x_n)_n$ is 2-equi-integrable and for every measurable set A:

$$\int_{\mathrm{A}} |x_n|.|y_n| ds \leq \sup_n \|y_n\|_2 \sup_n \|x_n 1_{\mathrm{A}}\|_2 .$$

Due to Lemma 6.13 the sequence $(|x_n|.|y_n|)_n$ strongly converges to the origin in $\mathrm{L}_1((0,1),\mathbb{R})$ and we get with $f^- = \inf(|s|.|t|,\ 1) = |f|$,

$$0 \leq \limsup_n \int_0^1 f^-(x_n(s), y_n(s)) ds = \limsup_n \int_0^1 |f|(x_n(s), y_n(s)) ds \leq \lim_n \int_0^1 |x_n|.|y_n| ds = 0 .$$

This proves that the sequence $(f(x_n, y_n))_n$ strongly converges in $\mathrm{L}_1((0,1),\mathbb{R})$ (and therefore the Ioffe criterion holds for f and $-f$). Thus I_f is sequentially strong weak continuous at $(0, y)$ because

$$\lim_n \int_0^1 f(x_n(s), y_n(s)) ds = \lim_n \int_0^1 f^-(x_n(s), y_n(s)) ds = 0 = \mathrm{I}_f(0, y) .$$

$\mathrm{I}_{f^{**}}$ is not semicontinuous at $(0, y)$: since $f^{**}(s,t) = 0$ if $s = 0$ and $f^{**}(s,t) = -1$ if $s \neq 0$ then

$$\mathrm{I}_{f^{**}}(0, y) = 0 > \liminf_n \mathrm{I}_{f^{**}}(n^{-1}, y) = -1 .$$

8 Lower compactness properties, usual examples.

The Banach space E is supposed separable. Hereafter $(f_n)_n$ is a sequence of extended real valued measurable integrands defined on $\Omega \times \mathrm{E}$.

Definition 8.1 *Let $\mathscr{X}$ be a set. A family $\mathscr{B}$ of subsets of $\mathscr{X}$ is said hereditary if for every element* X *of $\mathscr{B}$, every subset* Y *of* X *is an element of $\mathscr{B}$ too. A bornology on $\mathscr{X}$ is an hereditary family $\mathscr{B}$ of subsets of* X *stable by finite unions which covers* X.

When $(\mathscr{X}, \mathscr{T})$ is a locally convex topological space, the most usual bornologies are the family of $\mathscr{T}$-bounded sets it is called the Fréchet bornology, it is denoted by $\mathscr{B}_{\mathrm{F}}$; the family of sequentially $\mathscr{T}$-relatively compacts sets is called the $\mathscr{T}$-Hadamard bornology, it is denoted by $\mathscr{B}_{\mathrm{H}}$, and if more generally τ is a topology on $\mathscr{X}$ the family of τ-relatively sequentially compacts sets (denoted by $\mathscr{B}_\tau$) is a bornology. For more simplicity and clarity, in the sequel $\mathscr{X}$ will be a normed space of Orlicz type: more precisely a Lebesgue space endowed with its Fréchet bornology or its Hadamard bornologies associated to the strong or weak (star)topologies.

Definition 8.2 *Let $\mathscr{X}$ be a normed space contained in $\mathrm{L}_0(\Omega, \mathrm{E})$ endowed with a bornology $\mathscr{B}$. A sequence $(f_n)_n$ of extended real valued integrands defined on $\Omega \times \mathrm{E}$ is said to have the $\mathscr{B}$-lower compactness property (denoted $\mathscr{B}$-lcp) if for every element* X *of $\mathscr{B}$, every sequence $(x_n)_n$ of elements in* X *has the lower compactness property respect to $(f_n)_n$. The $\mathscr{B}_{\mathrm{F}}$-lcp is called Fréchet-lcp, the $\mathscr{B}_{\mathrm{H}}$-lcp is called Hadamard-lcp. Given a topology τ on $\mathscr{X}$, the $\mathscr{B}_\tau$-lcp is simply denoted by τ-lcp.*

Before to give concrete examples let us prove the following property.

Proposition 8.3 *Let a bornology $\mathscr{B}$ on a normed subspace $\mathscr{X}$ of* $L_0(\Omega, E)$*. A sequence $(f_n)_n$ of extended real valued integrands defined on $\Omega \times E$ has the $\mathscr{B}$-lcp if and only if each subsequence $(f_{n_k})_k$ has the $\mathscr{B}$-lcp. When τ is a topology on $\mathscr{X}$, a sequence $(f_n)_n$ of integrands has the $\mathscr{B}_\tau$-lcp if and only if every τ-converging sequence $(x_n)_n$ has the lcp respect to $(f_n)_n$.*

Proof of Proposition 8.3. Let X be an element of $\mathscr{B}$ and $(x_k)_k$ be a sequence of elements of X. Let a subsequence $(f_{n_k})_k$ of $(f_n)_n$. We want to prove that the sequence $(x_k)_k$ has the lcp respect to the sequence $(f_{n_k})_k$. Consider the sequence $(y_m)_m$ defined for $m \geq n_0$ by: $y_m = x_k$ if $n_k \leq m < n_{k+1}$. Since $(f_n)_n$ has the $\mathscr{B}$-lcp, then the sequence $(f_n^-(y_n))_n$ is uniformly integrable. But $(f_{n_k}^-(x_k))_k$ is a subsequence of $(f_n^-(y_n))_n$, therefore $(f_{n_k}^-(x_k))_k$ is uniformly integrable too. Now suppose that the sequence $(f_n)_n$ has not the $\mathscr{B}_\tau$-lcp there exists a subsequence $(f_{n_k})_k$ a τ-converging sequence $(x_k)_k$ such the sequence $(f_{n_k}^-(x_k))_k$ is not uniformly integrable. Define $y_m = x_k$ if $n_k \leq m < n_{k+1}$. Then $(y_m)_m$ τ-converges too and has not the lcp respect to $(f_m)_m$. The proof of Proposition 8.3 is complete. □

Given a Young integrand ϕ recall that the Orlicz space $L_\phi(\Omega, E, \mu) = \mathbb{R}_+ I_\phi^{\leq 1}$ becomes a decomposable normed space endowed with the Minkowski gauge associated to the sublevel set $I_\phi^{\leq 1}$, $\|x\|_\phi = \inf\{t > 0 : x \in t.I_\phi^{\leq 1}\}$. For $t > 0$, set $\phi_t(e) = \phi(te)$.

Proposition 8.4 *Let ϕ be a Young integrand and a sequence $(f_n)_n$ of measurable integrands such that:*
for every $\epsilon > 0$ and $\lambda > 0$ there exists a sequence $(u_n)_n$ of non negative uniformly integrable functions satisfying eventually

$$f_n \geq -\epsilon.\phi_\lambda(.) - u_n .$$

Then the sequence of integrands $(f_n)_n$ has the Fréchet-lcp on $L_\phi(\Omega, E)$.

Proof of Proposition 8.4 (When the measure is atomless, this condition is characteristic [30]). Let $(x_n)_n$ be a norm bounded sequence by m in $L_\phi(\Omega, E)$. Then $\sup_n I_\phi(m^{-1}x_n) \leq 1$. If the groth condition of Proposition 8.4 holds with $\lambda = m^{-1}$ then

$$\delta^+((f_n^-(x_n))_n) \leq \epsilon + \delta^+((u_n)_n) = \epsilon .$$

Therefore $\delta^+((f_n^-(x_n))_n) = 0$. The sequence $(f_n^-(x_n))_n$ being trivially bounded in $L_1(\Omega, \mathbb{R})$ it is, from Theorem 3.3 (a), eventually equi-integrable thus eventually uniformly integrable. We deduce that the sequence $(f_n)_n$ has the Fréchet-lcp on $L_\phi(\Omega, E)$. □

From the above result if $\mathscr{X} = L_p(\Omega, E)$ $1 \leq p < \infty$ and ϕ is the Young integrand $\phi(\omega, e) = p^{-1}\|e\|^p$, we obtain:

Corollary 8.5 *Let $1 \leq p < +\infty$ and a sequence $(f_n)_n$ of measurable integrands such that:*
for every $\epsilon > 0$ there exists a sequence $(u_n)_n$ of non negative uniformly integrable functions satisfying eventually

$$f_n \geq -\epsilon\|.\|^p - u_n .$$

Then the sequence of integrands $(f_n)_n$ has the Fréchet-lcp on $L_p(\Omega, E)$.

The case $L_\infty(\Omega, E)$ may be of interest:

Corollary 8.6 *Let $p = \infty$ and B be the unit ball of E. Let a sequence $(f_n)_n$ of measurable integrands verifying the property:*
For each $\lambda > 0$ there exists a uniformly integrable sequence of functions $(u_n)_n$ such that (eventually):

$$\inf_{e \in \lambda^{-1}B} f_n(., e) \geq -u_n .$$

Then the sequence of integrands $(f_n)_n$ has the Fréchet-lcp on $L_\infty(\Omega, E)$.

Proof of Corollary 8.6. $L_\infty(\Omega, E)$ is the Orlicz space associated to the Young integrand: $\phi(\omega, e) = \iota_B(e)$ for which $\phi_\lambda(\omega, e) = \iota_{\lambda^{-1}B}(e)$ $\square$.

Proposition 8.7 *Let ϕ be a Young integrand and a sequence $(f_n)_n$ of measurable integrands such that:*
for every $x \in L_\phi(\Omega, E)$, $\epsilon > 0$ there exist $\lambda > 0$, a positive constant $c \geq 1$ a sequence $(u_n)_n$ of non negative bounded integrable functions satisfying $\delta^+((u_n)_n) < \epsilon$ and eventually

$$f_n \geq -c.\phi_\lambda(. - x) - u_n .$$

Then the sequence of integrands $(f_n)_n$ has the (strong) Hadamard-lcp at $x \in L_\phi(\Omega, E)$.

Proof of Proposition 8.7. Let $(x_n)_n$ be a norm converging sequence to $x \in L_\phi(\Omega, E)$. If the growth condition of Proposition 8.7 holds there exists positive constants $\lambda, c > 1$ and a bounded sequence of integrable functions $(u_n)_n$ with $\delta^+((u_n)_n) < \epsilon$ satisfying eventually:

$$f_n \geq -c.\phi_\lambda(. - x) - u_n .$$

Therefore eventually:

$$f_n(x_n)^- \leq \frac{1}{2} c\phi_\lambda(2(x_n - x)) + u_n .(*)$$

Since $(x_n)_n$ converges to x, we have for n large enough

$$\|x_n - x\|_\phi \leq (2\lambda\epsilon^{-1}c)^{-1} \text{ , or eventually } I_\phi(\lambda(\epsilon^{-1}c)2(x_n - x)) = I_{\phi_\lambda}((\epsilon^{-1}c)2(x_n - x)) \leq 1$$

then eventually:

$$I_{\phi_\lambda}(2(x_n - x)) = I_{\phi_\lambda}(\epsilon c^{-1}\epsilon^{-1}c)2(x_n - x)) \leq \epsilon c^{-1} I_{\phi_\lambda}((\epsilon^{-1}c)2(x_n - x)) \leq \epsilon c^{-1} .$$

From $(*)$ we get the boundedness of the sequence $(f_n^-(x_n))_n$ in $L_1(\Omega, \mathbb{R})$ and:

$$\delta^+((f_n^-(x_n))_n) \leq \frac{1}{2} c I_{\phi_\lambda}(2(x_n - x)) + \delta^+((u_n)_n)) \leq \frac{3}{2}\epsilon .$$

This last inequality being valid for every $\epsilon > 0$ then from Theorem 3.3 (a) we obtain the equi-integrability of the sequence $(f_n^-(x_n))_n$, thus the uniform integrability of this sequence. $\square$

Corollary 8.8 *Let $1 \le p < +\infty$ and a sequence $(f_n)_n$ of measurable integrands verifying the property:*
For each $\epsilon > 0$ there exists a positive constant c_ϵ and a non negative bounded sequence of integrable functions $(u_n)_n$ with $\delta^+((u_n)_n) < \epsilon$ and such that (eventually):

$$f_n \ge -c_\epsilon \|.\|^p - u_n \,.$$

Then the sequence of integrands $(f_n)_n$ has the Hadamard-lcp on $\mathrm{L}_p(\Omega, \mathrm{E})$.

Proof of Corollary 8.8. For every $x \in \mathrm{L}_p(\Omega, \mathrm{E})$ use Proposition 8.7 and the following inequality: $\|e\|^p \le 2^{p-1}(\|e - x\|^p + \|x\|^p)$.

Corollary 8.9 *Let $p = \infty$. Let a sequence $(f_n)_n$ of measurable integrands verifying the property:*
For every $x \in \mathrm{L}_\infty(\Omega, \mathrm{E})$, $\epsilon > 0$, there exist $\lambda > 0$, a bounded sequence of integrable functions $(u_n)_n$ such that $\delta^+((u_n)_n) < \epsilon$ and eventually:

$$\inf_{e \in x + \lambda^{-1}\mathrm{B}} f_n(.,e) \ge -u_n \,.$$

Then the sequence of integrands $(f_n)_n$ has the strong Hadamard-lcp on $\mathrm{L}_\infty(\Omega, \mathrm{E})$.

Proof of Corollary 8.9. It is a consequence of Proposition 8.7, remark that $\mathrm{L}_\infty(\Omega, \mathrm{E})$ is the Orlicz space associated to the Young integrand: $\phi(\omega, e) = \iota_\mathrm{B}(e)$ for which $\phi_\lambda(\omega, e) = \iota_{\lambda^{-1}\mathrm{B}}(e)$. $\square$

In order to give another class of examples of sequences of integrands with the lower compactness property, let us consider a measurable integrand $f : \Omega \times \mathrm{E} \to \overline{\mathbb{R}}$. We will use its differential quotient defined by the formula:

$$[f](\omega, e_0, e, r) = [f_\omega](e_0, e, r) = r^{-1}(f_\omega(e_0 + re) - f_\omega(e_0))$$

If as usually, I_f is the integral functional associated to the integrand f and if x_0 is such that $f(x_0)$ is integrable remark that for every $x \in \mathrm{L}_0(\Omega, \mathrm{E})$ we have the equality:

$$[\mathrm{I}_f](x_0, x, r) = \mathrm{I}_{[f](x_0,.,r)}(x) \,.$$

Definition 8.10 *Given a Bornology $\mathscr{B}$ on a normed subspace $\mathscr{X}$ of $\mathrm{L}_0(\Omega, \mathrm{E})$, an integrand f is said to have the $\mathscr{B}$-differential lower compactness property on $\mathscr{X}$ (denoted $\mathscr{B}$-dlcp) at x_0 if for every sequence $(r_n)_n$ of positive numbers converging to 0, the sequence of differential quotients integrands $([f](x_0,.,r_n))_n$ has the $\mathscr{B}$-lcp. If $\mathscr{B}$ is the Fréchet-Bornology (respectively the Hadamard bornology, respectively the bornology associated to a topology τ) we call this notion Fréchet-differential lower compactness property (respectively Hadamard-dlcp, respectively τ-dlcp) at x_0.*

Proposition 8.11 *Let ϕ be a Young integrand and f be a measurable integrand satisfying the following condition:*
for every $\epsilon > 0$, and every $\lambda > 0$, there exists a family of non negative eventually integrable functions $\{u_r, r \in (0,1)\}$ uniformly integrable in $\mathrm{L}_1(\Omega, \mathbb{R})$ and verifying eventually

$$[f](x_0,.,r) \ge -\epsilon\phi_\lambda - u_r \,.$$

Then f has the Fréchet differential lcp on $\mathrm{L}_\phi(\Omega, \mathrm{E})$ at $x_0 \in \mathrm{L}_\phi(\Omega, \mathrm{E})$.

Proof. Let $(r_n)_n$ be a sequence of positive numbers converging to 0, using Proposition 8.4 with $f_n = [f](x_0,.,r_n)$ we deduce that the sequence of integrands $(f_n)_n$ has the Fréchet-lcp on $\mathrm{L}_\phi(\Omega,\mathrm{E})$. The result being valid for every sequence of positive numbers converging to 0, f has the Fréchet-lcp on $\mathrm{L}_\phi(\Omega,\mathrm{E})$. $\square$

Given a Young integrand, and an integrand f, we will use the following assumption:

$(\mathscr{S}_{\phi,x_0})$ $x_0 \in \mathrm{L}_\phi(\Omega,\mathrm{E})$, *for almost every* $\omega \in \Omega$, *the function* f_ω *is Lipschitzian on every ball of* E*; there exist two positive constants* c *and* λ_0 *such that for every* $\lambda > \lambda_0$ *there exist* $\beta > 0$ *and an integrable function* $u_\lambda \in \mathrm{L}_1(\Omega,\mathbb{R})$ *verifying:*

$$\sup_{e^* \in \partial^{\mathrm{C}} f(x_0+e)} \phi^*(\lambda e^*) \le c\phi(\beta e) + u_\lambda\,.$$

Where ∂^{C} *is the Clarke subdifferential see [13].* $(\mathscr{S}_\phi)$ *denotes* $(\mathscr{S}_{\phi,0})$.

Proposition 8.12 *Let* ϕ *be a Young integrand and* f *be a measurable integrand satisfying the condition* $(\mathscr{S}_{\phi,x_0})$. *Then* f *has the Fréchet differential lcp on* $\mathrm{L}_\phi(\Omega,\mathrm{E})$ *at* $x_0 \in \mathrm{L}_\phi(\Omega,\mathrm{E})$.

Proof. This is a consequence of Proposition 8.11 and the following Lemma with $e' = 0$:

Lemma 8.13 *A measurable integrand* $f : \Omega \times \mathrm{E} \to \mathbb{R}$ *satisfying* $(\mathscr{S}_{\phi,x_0})$ *verifies also: for every* $\epsilon > 0$ *and every* $\lambda > 0$, *there exist* $r_\epsilon > 0$ *an integrable function* $u_{\epsilon,\lambda}$ *such that for* $0 < r \le r_\epsilon$ *and* $e, e' :\in \mathrm{E}$

$$|r^{-1}(f(x_0+re) - f(x_0+re'))| \le \epsilon\phi_\lambda(e-e') + u_{\epsilon,\lambda}$$

Proof of Lemma 8.13. Let $\epsilon > 0$, $\lambda > 0$. From Lebourg's Mean value theorem [13], for each $e, e' \in \mathrm{E}$, $0 < r < 1$, for each $\omega \in \Omega$ there exists $0 < \theta_n(\omega) < 1$, $y = x_0 + \theta r(e-e')$ and $x^* \in \partial^{\mathrm{C}} f(y)$ such for every $\epsilon > 0$ and $\lambda > 0$:

$$r_n^{-1}(f(x_0+re) - f(x_0+re')) = \langle (\epsilon\lambda)^{-1}x^*, \epsilon\lambda(e-e')\rangle\,.$$

Therefore due to the Young inequality, and condition $(\mathscr{S}_\phi)$, when $(\epsilon\lambda)^{-1} \ge \lambda_0$ that is when $\epsilon \le \lambda^{-1}\lambda_0^{-1}$, there exist $\beta > 0$ and an integrable function $u_{(\epsilon\lambda)^{-1}}$ verifying eventually:

$$|r_n^{-1}(f(x_0+re) - f(x_0+re'))| \le \phi^*((\epsilon\lambda)^{-1}x^*) + \phi(\epsilon\lambda(e-e'))\,,$$

$$|r_n^{-1}(f(x_0+re)-f(x_0+re'))| \le c\phi(\beta(\theta_n r(e-e')) + u_{(\epsilon\lambda)^{-1}} + \epsilon\phi_\lambda(e-e') \le cr^{\frac{1}{2}}\phi(\beta r^{\frac{1}{2}}e) + \epsilon\phi_\lambda(e) + u_{(\epsilon\lambda)^{-1}}$$

when we have eventually: $0 < cr^{\frac{1}{2}} \le \epsilon$ and $0 < \beta r^{\frac{1}{2}} \le \lambda$, we deduce:

$$|r^{-1}(f(x_0+re) - f(x_0+re'))| \le 2\epsilon\phi_\lambda + u_{(\epsilon\lambda)^{-1}}$$

This ends the proof of Lemma 8.13. $\square$

End of the proof of Proposition 8.12. Since f satisfies the condition of Lemma 8.13, it satisfies the growth condition of Proposition 8.11 therefore f has the Fréchet differential lcp on $\mathrm{L}_\phi(\Omega,\mathrm{E})$ at $x_0 \in \mathrm{L}_\phi(\Omega,\mathrm{E})$. The proof of Proposition 8.12 is complete. $\square$

Corollary 8.14 *A measurable integrand $f : \Omega \times \mathrm{E} \to \mathbb{R}$ satisfying the condition of Lemma 8.13 (or clearly the condition $(\mathscr{S}_{\phi,x_0})$) at $x_0 \in \mathrm{L}_\phi(\Omega,\mathrm{E})$ verifies too eventually:*

$$\sup_{\|x\|_\phi \le 1, \|y\|_\phi \le 1} |\mathrm{I}_f(x_0 + rx) - \mathrm{I}_f(x_0 + ry)| - mr\|x-y\|_\phi \le 0\,.$$

Proof. Let B_ϕ be the unit ball of $\mathrm{L}_\phi(\Omega,\mathrm{E})$, when f satisfies the condition of Lemma 8.13, for every $\lambda > 0$, there exist $r_1 > 0$ and $u_{1,\lambda^{-1}}$ such that for $0 < t < r_1$ and $x', y' \in \mathrm{L}_\phi(\Omega,\mathrm{E})$:

$$t^{-1}|(f(x_0 + tx') - f(x_0 + ty'))| \le \phi(\frac{1}{2}\lambda^{-1}(x' - y')) + u_{1,\frac{1}{2}\lambda^{-1}}\,.$$

Therefore for every $x, y \in \lambda\mathrm{B}_\phi$, taking $x' = \|x-y\|_\phi^{-1}x$ and $y' = \|x-y\|_\phi^{-1}y$ and $t = r\|x-y\|_\phi$ we get for $r \le r_1\lambda^{-1}$

$$\sup_{x,y\in\lambda\mathrm{B}_\phi} \|x-y\|_\phi^{-1} r^{-1}|(f(x_0+rx) - f(x_0+ry))| \le \phi(\frac{1}{2}(x'-y')) + u_{1,\frac{1}{2}\lambda^{-1}}$$

thus, for every $x, y \in \lambda\mathrm{B}_\phi$, and $r \le r_1\lambda^{-1}$,

$$\|x-y\|_\phi^{-1} r^{-1}|(f(x_0+rx) - f(x_0+ry))| \le \frac{1}{2}\phi(\frac{(x-y)}{\|x-y\|_\phi}) + u_{1,\frac{1}{2}\lambda^{-1}}\,.$$

thus by integration, for every $x, y \in \lambda\mathrm{B}_\phi$, and $r \le r_1\lambda^{-1}$,

$$\|x-y\|_\phi^{-1} r^{-1}|(\mathrm{I}_f(x_0+rx) - \mathrm{I}_f(x_0+ry)| \le \frac{1}{2} + \int_\Omega u_{1,\frac{1}{2}\lambda^{-1}}\,d\mu = m < \infty$$

therefore for $r \le r_1\lambda^{-1}$:

$$\sup_{x,y\in\in\lambda\mathrm{B}_\phi} r^{-1}|(\mathrm{I}_f(x_0+rx) - \mathrm{I}_f(x_0+ry))| - \le m\|x-y\|_\phi\,,$$

this proves that with $\eta = r_1\lambda^{-1}$ and $r \le \eta$ we have:

$$\sup_{x,y\in\lambda\mathrm{B}_\phi} |\mathrm{I}_f(x_0+rx) - \mathrm{I}_f(x_0+ry)| - mr\|x-y\|_\phi \le 0\,. \square$$

Proposition 8.15 *Let ϕ be a Young integrand and f be an integrand verifying $(\mathscr{S}_\phi)$, then the condition $(\mathscr{S}_{\phi,x_0})$ holds at every point $x_0 \in \mathrm{E}_\phi(\Omega,\mathrm{E}) = \{x \in \mathrm{L}_\phi(\Omega,\mathrm{E}) : \forall\lambda > 0, \phi(\lambda x) \in \mathrm{L}_1(\Omega,\mathbb{R},\mu)\}$.*

Proof. Suppose that $x_0 \in \mathrm{E}_\phi(\Omega,\mathrm{E})$ and that the integrand f verifies $(\mathscr{S}_\phi)$. Then for every $\lambda > \lambda_0$:

$$\sup_{e^*\in\partial^{\mathrm{C}} f(x_0+e)} \phi^*(\lambda e^*) \le c\phi(\beta(x_0+e)) + u_\lambda \le \frac{1}{2}c\phi(2\beta e) + \frac{1}{2}c\phi(2\beta x_0) + u_\lambda\,.$$

Since $x_0 \in \mathrm{E}_\phi(\Omega,\mathrm{E})$, then $\phi(2\beta x_0)$ is integrable, this proves that the condition $(\mathscr{S}_{\phi,x_0})$ of Proposition 8.12 is true at x_0. $\square$

For more clarity and simplicity, in the sequel of this section $\mathscr{X} = \mathrm{L}_p(\Omega, \mathrm{E})$, $p^{-1} + q^{-1} = 1$. Recall that when $1 < p < \infty$ and $\phi = p^{-1}\|.\|^p$, then $\phi^* = q^{-1}\|.\|^q$, when $\phi = \|.\|$, then $\phi^* = \iota_{\mathrm{B}^*}$ where B^* is the unit ball of E^*. We will made the following assumptions on f:

($\mathscr{S}_p$)*:* $1 \le p < \infty$ *and there exist a positive constant* c *and a* q*-integrable non negative function* a *such that for every* $e \in \mathrm{E}$*, and* $e^* \in \partial^{\mathrm{C}} f_\omega(e)$*,*

$$\|e^*\|_* \le c\|e\|^{p-1} + a\,.$$

($\mathscr{S}_\infty$) *There exist* $\eta > 0$ *and an integrable function* c *such that:*

$$\|e\| \le \eta \Rightarrow f_\omega(x_0(\omega) + e) - f_\omega(x_0(\omega)) \ge -c(\omega)\|e\|\,.$$

Corollary 8.16 *Suppose* $1 < p < \infty$ *and* $\phi = p^{-1}\|.\|^p$*. Let a measurable integrand* $f : \Omega \times \mathrm{E} \to \mathbb{R}$*. The following assertions are equivalent:*
(a) *for every point* $x_0 \in \mathrm{L}_p(\Omega, \mathrm{E})$ *the integrand* f *verifies* ($\mathscr{S}_{\phi, x_0}$)*,*
(b) *there exists a point* $x_0 \in \mathrm{L}_p(\Omega, \mathrm{E})$ *such that* f *verifies* ($\mathscr{S}_{\phi, x_0}$)*,*
(c) *the integrand* f *verifies* ($\mathscr{S}_p$)*.*
Moreover when they hold, the integrand f *has the Fréchet-dlcp at every* $x_0 \in \mathrm{L}_p(\Omega, \mathrm{E})$*.*

Proof of Corollary 8.16. First let us proves that (b) $\Rightarrow$ (c). Suppose that f verifies ($\mathscr{S}_{\phi, x_0}$) holds with $\phi = p^{-1}\|.\|^p$ and the point x_0 is in $\mathrm{L}_p(\Omega, \mathrm{E})$. For almost every $\omega \in \Omega$, the function f_ω is Lipschitzian on every ball of E, moreover keeping $\lambda = \lambda_0$ in the definition of ($\mathscr{S}_{\phi, x_0}$), we obtain the existence of a positive constant c and an element u of $\mathrm{L}_1(\Omega, \mathbb{R})$ such that for every $e \in \mathrm{E}$, and $e^* \in \partial^{\mathrm{C}} f_\omega(x_0 + e)$,

$$\|e^*\|_*^q \le c\|e\|^p + u \le 2^{p-1}c(\|x_0 + e\|^p + \|x_0\|^p) + u\,.$$

Therefore there exist a positive constant d, an integrable function v such that

$$\|e^*\|_*^q \le d\|x_0 + e\|^p + v$$

since for every $0 < s < 1$, $t \to t^s$ is subadditive, with $s = q^{-1}$, we get for every $e^* \in \partial^{\mathrm{C}} f_\omega(x_0 + e)$,

$$\|e^*\|_* \le d^s\|x_0 + e\|^{ps} + v^s\,,$$

but $ps = p - 1$, the function v^s being q-integrable we obtain for every $e^* \in \partial^{\mathrm{C}} f_\omega(e)$,

$$\|e^*\|_* \le d^s\|e\|^{p-1} + v^s\,.$$

This proves that the integrand f verifies ($\mathscr{S}_p$) and proves that (b) $\Rightarrow$ (c). Now suppose that the integrand f verifies ($\mathscr{S}_p$). Then there exists a positive constant c a non negative q-integrable function a such for $e^* \in \partial^{\mathrm{C}} f_\omega(e)$:

$$\|e^*\|_* \le c\|e\|^{p-1} + a\,,$$

therefore since $\|e + e^{'}\|^q \le 2^{q-1}(\|e\|^q + \|e\|^q)$, we deduce for $e^* \in \partial^{\mathrm{C}} f_\omega(e)$:

$$\|e^*\|_*^q = (c\|e\|^{p-1} + a)^q \le 2^{q-1}(c^q(\|e\|^{q(p-1)} + a^q) = 2^{q-1}c^q\|e\|^{p)} + 2^{q-1}a^q\,,$$

thus we obtain the existence of a positive real number $d > 0$ and an integrable function u such for every $e^* \in \partial^{\mathrm{C}} f_\omega(e)$:

$$\|e^*\|_*^q \le d\|e\|^p + u\,.$$

Then for every point x_0 is in $\mathrm{L}_p(\Omega,\mathrm{E})$, for every $e^* \in \partial^{\mathrm{C}} f_\omega(e)$:

$$\|e^*\|_*^q \le d2^{p-1}(\|x_0+e\|^p + \|x_0\|^p) + u)\,,$$

but for every $\lambda > 0$ for every $e^* \in \partial^{\mathrm{C}} f_\omega(e)$, with the above inequality we reach:

$$\|\lambda e^*\|_*^q \le d2^{p-1}\|\lambda^{qp^{-1}} e\|^p + \lambda^q(d2^{p-1}\|x_0\|^p + u)\,.$$

This proves that the integrand f verifies $(\mathscr{S}_{\phi,x_0})$ at any point x_0 in $\mathrm{L}_p(\Omega,\mathrm{E})$, with $\phi = p^{-1}\|.\|^p$, and $(c) \Rightarrow (a)$. The proof of the last assertion is an immediate consequence of Proposition 8.12,$\square$

Let us consider the case $p = \infty$.

Corollary 8.17 *If the measurable integrand f satisfies the condition $(\mathscr{S}_\infty)$ then the integrand f has the Fréchet-dlcp on $\mathrm{L}_\infty(\Omega,\mathrm{E})$ at the point $x_0 \in \mathrm{L}_\infty(\Omega,\mathrm{E})$.*

Proof of Corollary 8.17. Given $\lambda > 0$, and $r > 0$ such $\lambda^{-1} r < \eta$, then:

$$\|e\| \le \lambda^{-1} \Rightarrow \|e\| r \le \lambda^{-1} r < \eta, \text{ and: } [f](x_0,e,r) \ge -\lambda^{-1} c$$

or equivalently the property of Proposition 8.11 holds with $\phi = \iota_{\mathrm{B}}$, for every $\epsilon > 0$ and $\lambda > 0$ we have eventually:

$$[f](x_0,e,r) \ge -\epsilon\phi_\lambda(e) - \lambda^{-1} c\,.\square$$

Hereafter $\mathrm{L}_p(\Omega,\mathrm{E})$ is endowed with the topology $\sigma_p = \sigma(\mathrm{L}_p(\Omega,\mathrm{E})\mathrm{L}_q(\Omega,\mathrm{E}^*))$ $p^{-1} + q^{-1} = 1$.

Proposition 8.18 *Let $1 < p \le \infty$, the Banach E being reflexive. When $p = \infty$, suppose $\mathrm{L}_1(\Omega,\mathrm{E}^*)$ is separable. Let $x_0 \in \mathrm{L}_p(\Omega,\mathrm{E})$ with $f(x_0)$ integrable and let us consider the following assertions:*
(a) the integrand f has the σ_p-dlcp at x_0,
(b) the integrand f has the Fréchet-dlcp at x_0
(c) for every sequence $(r_n)_n$ of positive real numbers converging to 0 the sequence of differential quotients $([f](x_0,.,r_n))_n$ satisfies the σ_p-Ioffe's criterion (see Definition 6.6) at every point of $\mathrm{L}_p(\Omega,\mathrm{E})$.
Then $(a) \Rightarrow (b) \Rightarrow (c)$. Moreover if the measure μ is atomless all the assertions are equivalent.

Proof of Proposition 8.18. First remark if $1 < p < \infty$ since E is reflexive, then $\mathrm{L}_p(\Omega,\mathrm{E})$ is reflexive too. Therefore the bounded sets of $\mathrm{L}_p(\Omega,\mathrm{E})$ are the relatively sequentially σ_p-compacts sets. When $p = \infty$, since $\mathrm{L}_1(\Omega,\mathrm{E}^*)$ is separable, the bounded sets of $\mathrm{L}_\infty(\Omega,\mathrm{E})$ are the relatively sequentially σ_∞-compacts sets. Therefore from Proposition 8.3, $(a) \Leftrightarrow (b)$. Then always $(a) \Leftrightarrow (b) \Rightarrow (c)$. Conversely, let us prove that $(c) \Rightarrow (b)$. If (b) fails there exist a sequence $(r_n)_n$ of positive real numbers converging to 0, and a bounded sequence $(x_n)_n$ such the sequence $([f]^-(x_0,x_n,r_n))_n$ is not uniformly integrable. Since the measure μ is atomless from

Theorem 3.3 (b), due to the definition of the index of equi-integrability Definition 3.2, there exists a decreasing sequence $(A_k)_k$ of measurable sets with a negligible intersection and a σ_p-converging subsequence $(x_{n_k})_k$ satisfying:

$$\lim_k \int_{A_k} [f]^-(x_0, x_{n_k}, r_{n_k}) d\mu > \epsilon > 0 .$$

Define $B_k = A_k \cap \{[f](x_0, x_{n_k}, r_{n_k}) \leq 0\}$. Then since $[f](x_0, 0, r_n) = 0$,

$$[f]^-(x_0, x_{n_k}, r_{n_k}) 1_{A_k} = -[f](x_0, x_{n_k}, r_{n_k}) 1_{B_k} = -[f](x_0, x_{n_k} 1_{B_k}, r_{n_k}) \text{ , and}$$

$$\int_{A_k} [f]^-(x_0, x_{n_k} 1_{B_k}, r_{n_k}) d\mu = \int_{A_k} [f]^-(x_0, x_{n_k}, r_{n_k}) d\mu .$$

But $1 < p \leq \infty$ thus $1 \leq q < \infty$, and since $(A_k)_k$ has a negligible intersection, the sequence $(x_{n_k} 1_{B_k})_k$ σ_p-converges to the origin and verifies for every integer k:

$$I_{[f](x_0,.,r_{n_k})}(x_{n_k} 1_{B_k}) \leq 0 \text{ , and}$$

$$\lim_k \int_{A_k} [f]^-(x_0, x_{n_k} 1_{B_k}, r_{n_k}) = \lim_k \int_{A_k} [f]^-(x_0, x_{n_k}, r_{n_k}) d\mu > \epsilon > 0 .$$

This proves with Theorem 3.3 (b), that the σ_p-Ioffe's criterion for the sequence $([f](x_0, ., r_{n_k}))_n$ fails at $x = 0$. We have proved the desired equivalences. □

9 Solid and integral Bornologies, subdifferentials

In this short section, we provide statements and proofs valid in a more general setting than the Lebesgue spaces. Let $(\mathscr{X}, \|.\|)$ be a normed space. Recall that the differential quotient $[f]$ of $f \in \overline{\mathbb{R}}^{\mathscr{X}}$ at a point x_0 of its domain is defined by the formula

$$[f](x_0, x, r) = r^{-1}(f(x_0 + rx) - f(x_0)) .$$

Observe that for all $x^* \in \mathscr{X}^*$ we have: $[f - \langle x^*, .\rangle] = [f] - \langle x^*, .\rangle$. Following the presentation given in [52] section 4.1.6, for a function $f \in \mathbb{R}^{\mathscr{X}}$ finite at x_0, the subdifferential associated to a bornology $\mathscr{B}$ on $\mathscr{X}$ is the set $\partial^{\mathscr{B}} f(x_0)$ of $x^* \in \mathscr{X}^*$ such that for all $X \in \mathscr{B}$ one has:

$$\liminf_{r \to 0_+} \inf_{x \in X} ([f](x_0, x, r) - \langle x^*, x\rangle) \geq 0 ,$$

or equivalently for all $X \in \mathscr{B}$:

$$\limsup_{r \to 0_+} \sup_{x \in X} ([f](x_0, x, r) - \langle x^*, x\rangle)^- = 0 .$$

The elements of $\partial^{\mathscr{B}} f(x_0)$ are called the $\mathscr{B}$-subderivatives of f at x_0.
If $\mathscr{B}$ is the bounded sets of $\mathscr{X}$ then $\partial^{\mathscr{B}} f(x_0)$ is the Fréchet subdifferential $\partial^F f(x_0)$ of f at x_0.
Classically an equivalent definition is the following:

$$\partial^F f(x_0) = \{x^* \in \mathscr{X}^* : \liminf_{x \to x_0} \frac{f(x) - f(x_0) - \langle x^*, x - x_0\rangle}{\|x - x_0\|} \geq 0\}.$$

When $\partial^{F} f(x_0)$ is non empty, the function f is said to be Fréchet subdifferentiable at the point $x_0 \in X$ of its domain.
If $\mathscr{T}$ is a topology on $\mathscr{X}$ finer or equal than the weak topology of $\mathscr{X}$ and $\mathscr{B}$ is the bornology associated to the sequentially relatively compact sets, then $\partial^{\mathscr{B}} f(x_0)$ is the Hadamard subdifferential $\partial^{\mathscr{T}} f(x_0)$ of f at x_0. One can see that an equivalent definition is the following:

$$\partial^{\mathscr{T}} f(x_0) = \{x^* \in \mathscr{X}^* : \forall x \in X, \langle x^*, x\rangle \le f^{\mathscr{T}}(x_0; x)\},$$

where $f^{\mathscr{T}}(x_0; .)$ is the sequential Hadamard directional subderivate of f at x_0 defined by:

$$f^{\mathscr{T}}(x_0; x) = \inf_{(x_n) \xrightarrow{\mathscr{T}} x,\, (r_n) \to 0_+} \liminf_n [f](x_0, x_n, r_n) = \inf_{(r_n) \to 0_+} seq\mathscr{T} - li_e[f](x_0, ., r_n)(x).$$

In this section, E is a separable Banach space and $\mathscr{X}$ will be a decomposable normed subspace of $L_0(\Omega, E)$ and sometimes the following (Rockafellar's decomposability) property is used:

($\mathscr{D}$) *There exists an increasing covering* $(\Omega_n)_n$ *of* Ω *such that for every integer* n,

$$L_\infty(\Omega, E) 1_{\Omega_n} = \{y 1_{\Omega_n}, y \in L_\infty(\Omega, E)\} \subset \mathscr{X}.$$

Definition 9.1 *A bornology* $\mathscr{B}$ *on* $\mathscr{X}$ *is said solid if for every element* $X \in \mathscr{B}$, *the subset* $\mathbb{T}(X) = \{x 1_A : x \in X, A \in \mathbb{T}\}$ *is an element of* $\mathscr{B}$.

Since for every $A \in \mathbb{T}$ and $x \in L_p(\Omega, E)$ we have $\|x 1_A\|_p \le \|x\|_p$, we deduce that the Fréchet bornology on a Lebesgue space $L_p(\Omega, E)$ is solid. More generally, if $\phi : \Omega \times E \to \overline{\mathbb{R}}_+$ is a Young integrand, the associated Orlicz space $\mathbb{R}_+ I_\phi^{\le 1} = L_\phi(\Omega, E, \mu)$ generated by the sublevel set $I_\phi^{\le 1}$ becomes a decomposable normed space endowed with the Minkowski gauge associated to the sublevel set $I_\phi^{\le 1}$, $\|x\|_\phi = \inf\{t > 0 : x \in t.I_\phi^{\le 1}\}$. Moreover for every $A \in \mathbb{T}$ and $x \in L_\phi(\Omega, E)$ we have $\|x 1_A\|_\phi \le \|x\|_\phi$; therefore the Fréchet bornology on an Orlicz space is solid. In addition, it verifies always the property ($\mathscr{D}$) [23]. When E is reflexive, due to the weak compactness Dunfort-Pettis's criterion , the bornology on $L_1(\Omega, E)$ associated to the weakly compact sets is solid.

Theorem 9.2 *Given a decomposable normed subspace* $\mathscr{X}$ *of* $L_0(\Omega, E)$ *and* f *a measurable integrand, if* $\mathscr{B}$ *is a solid bornology on* $\mathscr{X}$, *and* $x^* \in \mathscr{X}^* \cap L_0(\Omega, E^*_{\sigma^*})$ *then the following assertions are equivalent:*
(a) x^* *is a* $\mathscr{B}$*-subderivative of* I_f *at* x_0.
(b) *for every element* $B \in \mathscr{B}$, $\lim_{r \to 0_+} \sup_{x \in B} \int_\Omega [f - \langle x^*, .\rangle]^-(x_0, x, r) = 0$.

Proof of Theorem 9.2. Keeping the integrand $f - \langle x^*, .\rangle$ it suffices to consider the case $x^* = 0$. If (b) fails, there exist $\epsilon > 0$, a sequence $(r_n)_n$ of positive real numbers converging to 0, an element $X \in \mathscr{B}$ a sequence $(x_n)_n$ in X, such eventually $\int_\Omega [f]^-(x_0, x_n, r_n) d\mu \ge \epsilon > 0$. For each integer n there exists a measurable set $A_n = \{[f](x_0, x_n, r_n) \le 0\}$ such that:

$$\int_\Omega [f]^-(x_0, x_n, r_n) d\mu = -\int_{A_n} [f](x_0, x_n, r_n) d\mu\,.$$

But since $[f](x_0, 0, r_n) = 0$,

$$[f](x_0, x_n, r_n)1_{A_n} = [f](x_0, x_n 1_{A_n}, r_n)\,,$$

we get eventually:

$$\int_\Omega [f](x_0, x_n 1_{A_n}, r_n)\,d\mu = -\int_\Omega [f]^-(x_0, x_n, r_n)\,d\mu \le -\epsilon\,.$$

The sequence $(x_n)_n$ being in X, since $\mathscr{B}$ is solid, the sequence $(x_n 1_{A_n})_n$ is in $\mathbb{T}(X) \in \mathscr{B}$. Thus the above inequality shows that there exists an element $Y = \mathbb{T}(X) \in \mathscr{B}$ with

$$\liminf_{r\to 0_+} \inf_{x\in Y} [I_f](x_0, x, r) \le -\epsilon\,.$$

Therefore 0 is not a $\mathscr{B}$-subderivative of I_f at x_0. We have proved that $(a) \Rightarrow (b)$.
Conversely suppose (b) holds. Given an element $X \in \mathscr{B}$ we get:

$$\liminf_{r\to 0_+} \inf_{x\in X} ([I_f - \langle x^*, .\rangle](x_0, x, r) \ge -\limsup_{r\to 0_+} \sup_{x\in X} \int_\Omega [f - \langle x^*, .\rangle]^-(x_0, x, r)\,d\mu = 0\,.$$

This proves that x^* is a $\mathscr{B}$-subderivative of I_f at x_0 and $(b) \Rightarrow (a)$. □

Definition 9.3 *An integral bornology $\mathscr{B}$ on $\mathscr{X}$ is a bornology such there exists a family $\mathbb{F} = \{\phi_i, i \in I\}$ of non negative integrands verifying $\phi_i(0) = 0$ for all $i \in I$ and such that $\mathscr{B}$ is the family of sets* B *contained in a sublevel set of an element of the family of integral functionals $\{I_{\phi_i}, i \in I\}$.*

Notice that any integral bornology is solid. The Fréchet bornology of $L_p(\Omega, E)$, $1 \le p < \infty$ is an integral bornology associated to the singleton $\mathbb{F} = \{\|.\|^p\}$. The Fréchet bornology of $L_\infty(\Omega, E)$, is an integral bornology associated to the family $\mathbb{F} = \{I_{\phi_\lambda},\ \lambda > 0\}$ where $\phi_\lambda(\omega, e) = \iota_{B_E}(\lambda e) = \iota_{\lambda^{-1}B_E}(e)$ and B_E is the unit ball of E. More generally, considering a Young integrand ϕ and the associated Orlicz space $L_\phi(\Omega, E)$ then the Fréchet bornology of $L_\phi(\Omega, E)$ is the family of sets B contained in a sublevel set of an element of the family of integral functionals $\{I_{\phi_\lambda},\ \lambda > 0\}$ where $\phi_\lambda(\omega, e) = \phi(\omega, \lambda e)$. Since for every weakly compact set X of $L_1(\Omega, E)$, due to the Dunfort-Pettis and de la Vallée Poussin criterions Theorem 3.4 (b), there exists an integrand ϕ of α-Young type such that $\sup_{x\in X} I_\phi(x) \le 1$, the bornology associated to the weakly relatively compact sets of $L_1(\Omega, E)$ called the weak Hadamard bornology is the integral bornology on $L_1(\Omega, E)$ associated to the family of α-Young integrands.

Theorem 9.4 *Let a decomposable normed subspace $\mathscr{X}$ of $L_0(\Omega, E)$ verifying in addition the decomposability property $(\mathscr{D})$, and let f be a measurable integrand. If $\mathscr{B}$ is an integral bornology on $\mathscr{X}$ associated to the family $\mathbb{F} = \{\phi_i, i \in I\}$, let us consider the following assertions:*
(a) $x^* \in \mathscr{X}^* \cap L_0(\Omega, E^*_{\sigma^*})$ is a $\mathscr{B}$-subderivative of I_f at x_0.
(b) for every $i \in I$, for every $\epsilon > 0$, there exists a family of non negative eventually integrable functions $\{u_r, r \in (0,1)\}$ norm converging to 0 in $L_1(\Omega, \mathbb{R})$ and verifying eventually

$$[f - \langle x^*, .\rangle](x_0, ., r) \ge -\epsilon\phi_i - u_r\,.$$

Then always $(b) \Rightarrow (a)$. If in addition the measure is atomless then these assertions are equivalent.

Proof of Theorem 9.4. Proof of the sufficiency part. Let $X \in \mathscr{B}$, there exists $(i,s) \in I \times \mathbb{R}_+$ such $\sup_{x\in X} I_{\phi_i}(x) \le s$. Using condition (b), for every $\epsilon > 0$, there exists a family of non negative eventually integrable functions $\{u_r, r \in (0,1)\}$ norm converging to 0 and verifying eventually for $x \in X$: $[f - \langle x^*, .\rangle]^-(x_0, x, r) \le \epsilon\phi_i(x) + u_r$, therefore for every $\epsilon > 0$:

$$\limsup_{r\to 0_+} \sup_{x\in X} \int_\Omega [f - \langle x^*, .\rangle]^-(x_0, x, r)\, d\mu \le \epsilon . s\,,$$

thus

$$\lim_{r\to 0_+} \sup_{x\in X} \int_\Omega [f - \langle x^*, .\rangle]^-(x_0, x, r)\, d\mu = 0\,,$$

and the sufficency part is proved with Theorem 9.2 (b). Conversely suppose that $x^* \in \mathscr{X}^* \cap L_0(\Omega, E^*_{\sigma^*})$ is a $\mathscr{B}$-subderivative of I_f at x_0. Then for every $i \in I$:

$$\limsup_{r\to 0_+} \sup_{I_{\phi_i}(x)\le 1; x\in\mathscr{X}} \int_\Omega [f - \langle x^*, .\rangle]^-(x_0, x, r)\, d\mu = 0\,.$$

Fix $i \in I$, and denotes ϕ_i by ϕ. Setting $g_r(\omega, e) = [f_\omega - \langle x^*(\omega), e\rangle]^-(x_0(\omega), e, r)$, we have:

$$\text{for every } \epsilon > 0 \text{ there exists } r_\epsilon > 0 : \sup_{0<r\le r_\epsilon} \sup_{I_\phi(x)\le 1; x\in\mathscr{X}} I_{g_r}(x) \le \epsilon.$$

The assumptions of [6] Corollary 5.7 with the constant multifunction $M(\omega) = E$ are satisfied. Indeed since $(\mathscr{D})$ holds $\mathscr{X}$ is rich (in sense of [6] Definition 3.7) in $S_M = L_0(\Omega, E)$. Using [6] Corollary 5.7, for every $0 < r < r_\epsilon$ we get the existence of a multiplier $y^*_r \ge 0$ such that the function

$$v_r(\omega) = \inf\{-g_r(\omega, e) + y^*_r \phi(\omega, e), e \in E\}$$

is integrable and verifies

$$-\epsilon + y^*_r \le \int_\Omega v_r\, d\mu\,.$$

Moreover since $\phi(0) = 0$ and $g_r(0) = 0$ the function v_r is non positive and we get for every $0 < r < r_\epsilon$ that $y^*_r \le \epsilon$ and that $\int_\Omega -v_r\, d\mu \le \epsilon - y^*_r \le \epsilon$. Define

$$u_{r,\epsilon}(\omega) = \inf\{-g_r(\omega, e) + \epsilon\phi(\omega, e), e \in E \ge v_r$$

then

$$g_r \le -v_r + y^*_r\phi \le -v_r + \epsilon\phi \le -u_{r,\epsilon} + \epsilon\phi\,.$$

We get: for $0 < r < r_\epsilon$, $u_{r,\epsilon} \ge 0$, $g_r \le \epsilon\phi_i + u_{r,\epsilon}$ and $\int_\Omega -u_{r,\epsilon}\, d\mu \le \epsilon$. Then $\lim_{r\to 0_+} \int_\Omega u_{r,\epsilon}\, d\mu = 0$. Indeed take $\epsilon > 0$. Given $0 < \epsilon' < \epsilon$, for $0 < r < r_{\epsilon'}$, we have $u_{r,\epsilon'} \le u_{r,\epsilon}$ and

$$\int_\Omega -u_{r,\epsilon}\, d\mu \le \int_\Omega -u_{r,\epsilon'}\, d\mu \le \epsilon'.$$

This proves that $\lim_{r\to 0} \int_\Omega u_{r,\epsilon}\, d\mu = 0$ and moreover $g_r \le \epsilon\phi - u_{r,\epsilon}$ $\square$

Taking the Fréchet bornology on an Orlicz space, we obtain immediatly the following criterion of subdifferentiability:

Corollary 9.5 *Let ϕ be a Young integrand and $f : \Omega \times \mathrm{E} \to \overline{\mathbb{R}}$ be a measurable integrand. Consider a function $x^* \in \mathrm{L}_{\phi^*}(\Omega, \mathrm{E}^*_{\sigma^*})$ and the following assertions:*
(a) x^* is a Fréchet subderivative of I_f at x_0 on $\mathrm{L}_\phi(\Omega, \mathrm{E})$,
(b) for every $0 < \epsilon$, $0 < \lambda$, there exists a family of non negative eventually integrable functions $(u_r)_{r>0}$ such that $\lim_{r\to 0} \| u_r \|_1 = 0$ and eventually:

$$[f - \langle x^*, .\rangle](x_0, ., r) \geq -\epsilon\phi_\lambda(.) - u_r \,.$$

Then always $(b) \Rightarrow (a)$. If in addition the measure is atomless then these assertions are equivalent.

The following particular consequence in case $1 < p < \infty$ is nothing else than a rephrase of the J. P Penot's characterization [51] Theorem 22. But not only, since it contains also the N. H. Chieu's and J. P Penot's characterization [29] Theorem and [51] Theorem 12 of the Fréchet subdifferential in case $p = 1$ (see Corollary 9.7).

Corollary 9.6 *Suppose $1 \leq p < \infty$ and $f : \Omega \times \mathrm{E} \to \overline{\mathbb{R}}$ be a measurable integrand. Given $x^* \in \mathrm{L}_q(\Omega, \mathrm{E}^*_{\sigma^*})$ let us consider the following assertions:*
(a) x^* is a Fréchet subderivative of I_f at x_0 on $\mathrm{L}_p(\Omega, \mathrm{E})$,
(b) For every $\epsilon > 0$ there exists a family of non negative eventually integrable functions $(u_r)_{r>0}$ such that $\lim_{r\to 0} \| u_r \|_1 = 0$ and eventually:

$$[f - \langle x^*, .\rangle](x_0, ., r) \geq -\epsilon \|.\|^p - u_r \,.$$

Then always $(b) \Rightarrow (a)$ and if addition the measure is atomless these assertions are equivalent.

Proof. The relations between the two assertions are an immediate consequence of Corollary 9.5 with $\phi = p^{-1}\|.\|^p$. □

Corollary 9.7 *Let $p = 1$, $f : \Omega \times \mathrm{E} \to \overline{\mathbb{R}}$ be a measurable integrand and suppose the measure is atomless. Given $x^* \in \mathrm{L}_\infty(\Omega, \mathrm{E}^*_{\sigma^*})$ the following assertions are equivalent:*
(a) x^* is a Fréchet subderivative of I_f at x_0 on $\mathrm{L}_1(\Omega, \mathrm{E})$,
(b) The integrand f verifies the assertion (b) of Corollary 9.5 with $p = 1$.
(c) x^* is a Moreau-Rockafellar subderivative of I_f at x_0 on $\mathrm{L}_1(\Omega, \mathrm{E})$.

Proof. First remark that it suffices to prove this Corollary in case $x^* = 0$. $(a) \Rightarrow (b)$ is a consequence of Corollary 9.5 with $\phi = \|.\|$. Now suppose that condition (b) of Corollary 9.5 holds with $x^* = 0$. Fix $\epsilon > 0$. For every $r > 0$, define $v_r = \inf_{e\in\mathrm{E}}[f](x_0, e, r) - \epsilon\|e\|$. Then v_r satisfies $-u_r \leq v_r \leq 0$, therefore the family $(v_r)_{r>0}$ of non positive eventually integrable functions norm converges to the origin of $\mathrm{L}_1(\Omega, \mathbb{R})$ when $r \to 0_+$. But

$$v_r = \inf_{e\in\mathrm{E}}[f](x_0, e, r)] - \epsilon\|e\| = \inf_{e\in\mathrm{E}} r^{-1}([](x_0, re) - \epsilon\|re\|) = r^{-1}v_1 \,.$$

Since the family $(v_r)_{r>0}$ of eventually integrable functions norm converges to the origin of $\mathrm{L}_1(\Omega, \mathbb{R})$ when $r \to 0_+$, neccesarily eventually $v_1 = 0 = v_r$. Therefore the integrand $[f]$ verifies eventually at x_0:

$$\inf_{e\in\mathrm{E}}[f](x_0, e, r) - \epsilon\|e\| \geq 0 \,.$$

Or equivalently,

$$\inf_{e\in E} f(x_0+e) - f(x_0) - \epsilon\|e\| \geq 0\,.$$

This last property being valid for every $\epsilon > 0$, one get for every $e \in E$, $f(x_0+e) - f(x_0) \geq 0$, therefore 0 is a Moreau-Rockafellar subderivative of I_f at x_0 on $L_1(\Omega, E)$. Thus $(b) \Rightarrow (c)$ is true. $(c) \Rightarrow (a)$ is immediate. □

In the same spirit let us consider the case $p = \infty$. Using Corollary 9.5 it can be obtained the following characteristic condition (when f is only measurable) when the measure is atomless; let us give a direct proof without this assumption but when f is normal.

Corollary 9.8 *Suppose $p = \infty$, and $f : \Omega \times E \to \overline{\mathbb{R}}$ is a normal integrand. An integrable function $x^* \in L_1(\Omega, E^*_{\sigma^*})$ is a Fréchet subderivative of I_f at x_0 on $L_\infty(\Omega, E)$, if and only if there exists a family of eventually non negative integrable functions $(u_r)_{r>0}$ such that $\lim_{r\to 0_+} \|u_r\|_1 = 0$ and eventually:*

$$\inf_{\|e\|\leq 1} [f - \langle x^*, .\rangle](x_0, e, r) \geq -u_r\,.$$

Proof of Corollary 9.8. Remark that it suffices to gives the proof in case $x^* = 0$. Let B_∞ be the unit ball of $L_\infty(\Omega, E)$. The condition is sufficient. Indeed:

$$\lim_{r\to +} \sup_{x\in B_\infty} \int_\Omega [f]^-(x_0, x, r)\, d\mu \leq \lim_{r\to 0_+} \int_\Omega u_r\, d\mu = 0\,,$$

then we obtain for $s > 0$ $\lim_{r\to 0_+} \sup_{x\in sB_\infty} \int_\Omega [f]^-(x_0, x, r)\, d\mu = 0$, this proves with Theorem 9.2 (b) that x^* is a Fréchet subderivative of I_f at x_0 on $L_\infty(\Omega, E)$. Conversely let us show the necessity. If 0 is a Fréchet subderivative of I_f at x_0 then with Theorem 9.2 (b) we get

$$\lim_{r\to 0_+} \sup_{x\in B_\infty} I_{[f]^-}(x_0, x, r) = 0\,.$$

But for every $r > 0$ if $u_r = \sup_{\|e\|\leq 1} [f]^-(x_0, e, r)$ then [31] Theorem 2.2 asserts that

$$\sup_{x\in B_\infty} I_{[f]^-}(x_0, x, r) = \int_\Omega u_r\, d\mu\,.$$

Therefore $\lim_{r\to 0_+} \int_\Omega u_r\, d\mu = 0$. The proof of Corollary 9.8 is complete. □

Using Theorem 9.4 one obtain the following criterion for the weak Hadamard subdifferentiability:

Corollary 9.9 *Suppose $p = 1$ and $f : \Omega \times E \to \overline{\mathbb{R}}$ is a measurable integrand. Given a function $x^* \in L_\infty(\Omega, E^*_{\sigma^*})$ let us consider the following assertions:*
(a) x^* is a weak Hadamard subderivative of I_f at x_0 on $L_1(\Omega, E)$:
(b) for every (respectively there exists an) integrable positive valued function α, for every α-Young integrand ϕ and for every $\epsilon > 0$ there exists a family of eventually non negative integrable functions $(u_r)_{r>0}$ such that $\lim_{r\to 0_+} \|u_r\|_1 = 0$ and eventually:

$$[f - \langle x^*, .\rangle](x_0, ., r) \geq -\epsilon\phi(.) - u_r\,.$$

Then always $(b) \Rightarrow (a)$ and if the measure is atomless these assertions are equivalent.

Proof. We have yet seen with De la Vallé-Poussin's Theorem 3.4 that for every positive valued integrable function α, the weak Hadamard bornology is the integral bornology associated with the family of α-Young integrands and the result is a consequence of Theorem 9.4. □

10 Additional results on Fréchet subdifferentiability

In this section, when E is separable, the study of some properties of the Fréchet subdifferentiability is mainly made in relation with the Fréchet differential compactness property.

Definition 10.1 *Let an integrand $f : \Omega \times \mathrm{E} \to \overline{\mathbb{R}}$, and $x_0 \in \mathrm{L}_0(\Omega, \mathrm{E})$. The integrand f is said Fréchet subdifferentiable along x_0 when for almost every $\omega \in \Omega$, the function f_ω is Fréchet subdifferentiable at $x_0(\omega)$. An element $x^* \in \mathrm{L}_0(\Omega, \mathrm{E}^*_{\sigma^*})$ is a Fréchet subderivative of f along x_0 when for almost every $\omega \in \Omega$, $x^*(\omega)$ is a Fréchet subderivative of f_ω at $x_0(\omega)$.*

The following result gives a practical sufficient criterion for the Fréchet subderivability of an integral functional.

Theorem 10.2 *Let us consider a decomposable normed space $(\mathscr{X}, \|.\|_{\mathscr{X}})$ topologically contained in some Lebesgue space $\mathrm{L}_p(\Omega, \mathrm{E}, \beta\mu)$ for some measurable positive valued function β. Suppose that the Fréchet bornology on $\mathscr{X}$ is solid. Let f be a measurable integrand and the following assertions:*
(a) $x^* \in \mathscr{X}^* \cap \mathrm{L}_0(\Omega, \mathrm{E}^*_{\sigma^*})$ is a Fréchet subderivative of I_f at x_0 on $\mathscr{X}$.
(b) The integrand $f - \langle x^*, \rangle$ has the Fréchet-dlcp at x_0.
Then $(a) \Rightarrow (b)$, if moreover x^* is a Fréchet subderivative of f along x_0, then $(b) \Rightarrow (a)$.

Proof of Theorem 10.2. Since the Fréchet bornology of $\mathscr{X}$ is solid then $(a) \Rightarrow (b)$ is a consequence of Theorem 9.2 in case where $\mathscr{B}$ is the Fréchet bornology. Conversely suppose (b) holds and x^* is a Fréchet subderivative of f along x_0. Setting $g(\omega, e) = f(\omega, x_0(\omega) + e) - f(x_0(\omega)) - \langle x^*(\omega), e\rangle$, one may suppose that $x^* = 0$ is a Fréchet subderivative of g along the origin, thus $g(\omega, e) = \|e\|\epsilon(\omega, e)$ with $\liminf_{e\to 0} \epsilon_\omega(e) \geq 0$, and that g has the Fréchet-dlcp. From Theorem 9.2 it suffices to prove that for every sequence $(r_n)_n$ of positive real numbers converging to 0, for every bounded sequence $(x_n)_n$ in $\mathscr{X}$, the sequence $([g]^-(0, x_n, r_n))_n$ strongly converges to the origin in $\mathrm{L}_1(\Omega, \mathbb{R})$.
The following Lemma is proved in [28]:

Lemma 10.3 *If 0 is a Fréchet subderivative of f along x_0, then the function $\epsilon(\omega, e) = \epsilon_\omega(e)$ if $e \neq 0$, $\epsilon(\omega, 0) = 0$, is measurable on $\Omega \times \mathrm{E}$, and verifies for every $(\omega, e) \in \Omega \times \mathrm{E}$: $f_\omega(x_0(\omega) + e) - f_\omega(x_0(\omega)) = \|e\|\epsilon(\omega, e)$. The integrand $\epsilon^- = -\min(\epsilon, 0)$ is measurable on $\Omega \times \mathrm{E}$ and moreover $\lim_{e\to 0} \epsilon^-(\omega, e) = 0$.*

From the preceding Lemma $[g]^-(0, x_n, r_n) = \|x_n\|\epsilon^-(r_n x_n)$, moreover since the sequence $(x_n)_n$ is bounded in $\mathscr{X}$ then by assumptions $(x_n)_n$ it is bounded in some $\mathrm{L}_p(\Omega, \mathrm{E}, \beta\mu)$ therefore the sequence $(r_n x_n)_n$ norm converges to 0 in $\mathrm{L}_p(\Omega, \mathrm{E}, \beta\mu)$, thus in $\beta\mu$- measure ([39] section 4.7 (or [58] Lemma 16.4)) and since β is positive valued, extracting subsequences almost everywhere converging to 0 we obtain that the convergence is in in local μ-measure , so is the convergence of the sequence $(\epsilon^-(r_n x_n)_n$. Let $v_n = \|x_n\|\epsilon(r_n x_n)$. Since f has the Fréchet-dlcp at x_0, the integrand g too at the origin. Therefore the sequence $(v_n^-)_n$ is uniformly integrable. We have $v_n^- = \|x_n\|\epsilon^-(r_n x_n)$ applying Lemma 6.13 with $\mathrm{E} = \mathbb{R}$, $y_n = \|x_n\|$, $u_n = \epsilon^-(r_n x_n)$, we deduce that $(v_n^-)_n$ converges strongly to 0 in $\mathrm{L}_1(\Omega, \mathbb{R}, \mu)$. The proof of Theorem 10.2 is complete. □

As a consequence of [22] Theorem 2.4 (vii), the statement of Theorem 10.2 is valid on every Orlicz space $\mathrm{L}_\phi(\Omega,\mathrm{E},\mu)$. Let us give a practical criterion for the Fréchet-subdifferentiability.

Corollary 10.4 *Let ϕ be a Young integrand, and f be a measurable integrand such that $x^* \in \mathrm{L}_{\phi^*}(\Omega,\mathrm{E}^*)$ is a Fréchet-subderivative of f along $x_0 \in \mathrm{L}_\phi(\Omega,\mathrm{E})$. Suppose that the following condition holds:*
for every $\epsilon > 0$, and every $\lambda > 0$, there exists a family of non negative eventually integrable functions $\{u_r, r \in (0,1)\}$ uniformly integrable in $\mathrm{L}_1(\Omega,\mathbb{R})$ and verifying eventually

$$[f](x_0,.,r) - \langle x^*,.\rangle \geq -\epsilon\phi_\lambda - u_r\,.$$

Then x^* is a Fréchet-subderivative of I_f at x_0 on $\mathrm{L}_\phi(\Omega,\mathrm{E})$.

Proof. due to Proposition 8.11, $f - \langle x^*,.\rangle$ has the Fréchet differential lcp on $\mathrm{L}_\phi(\Omega,\mathrm{E})$ at $x_0 \in \mathrm{L}_\phi(\Omega,\mathrm{E})$. The result is then an immediate consequence of Theorem 10.2. □

Given a Young function ϕ we will consider the following subspace of $\mathrm{L}_\phi(\Omega,\mathrm{E},\mu)$:

$$\mathrm{E}_\phi(\Omega,\mathrm{E}) = \{x \in \mathrm{L}_\phi(\Omega,\mathrm{E}) : \forall\lambda > 0, \phi(\lambda x) \in \mathrm{L}_1(\Omega,\mathbb{R},\mu)\}.$$

Given an E^*-valued multifunction M, $\mathrm{L}_{\phi^*}(\mathrm{M})_{\sigma^*}$, (respectively $\mathrm{E}_{\phi^*}(\mathrm{M})_{\sigma^*}$) denotes the set of almost everywhere selections of M which are in $\mathrm{L}_{\phi^*}(\Omega,\mathrm{E}^*_{\sigma^*})$ (respectively $\mathrm{E}_{\phi^*}(\Omega,\mathrm{E}^*_{\sigma^*})$).

Proposition 10.5 *Let ϕ be a Young integrand and f be an integrand such $x^* \in \mathrm{E}_{\phi^*}(\Omega,\mathrm{E}^*)$ is a Fréchet-subderivative of f along $x_0 \in \mathrm{L}_\phi(\Omega,\mathrm{E})$. If the condition $(\mathscr{S}_{\phi,x_0})$ of Theorem 8.12 holds then $\mathrm{E}_{\phi^*}(\partial^{\mathrm{F}} f(x_0))_{\sigma^*} \subset \partial^{\mathrm{F}}\mathrm{I}_f(x_0)$.*

Proof. First remark that for any $x^* \in \mathrm{E}_{\phi^*}(\Omega,\mathrm{E}^*)$ the integrand $f - \langle x^*,.\rangle$ has the Fréchet-dlcp at x_0. Indeed let f verifying $(\mathscr{S}_{\phi,x_0})$. Since for every $x^* \in \mathrm{E}_{\phi^*}(\Omega,\mathrm{E}^*)$, due to the Young inequality, for every $e \in \mathrm{E}$, and every $\lambda > 0$, $|\langle x^*, e\rangle| \leq \phi(\lambda e) + \phi^*(\lambda^{-1}x^*)$ with $\phi^*(\lambda^{-1}x^*)$ integrable, we obtain with Lemma 8.13, that $f - \langle x^*,.\rangle$ verifies the condition of Corollary 10.4. If moreover $x^* \in \mathrm{E}_{\phi^*}(\partial^{\mathrm{F}} f(x_0))_{\sigma^*}$ is a Fréchet-subderivative of f along x_0, then due to Corollary 10.4, x^* is a Fréchet-subderivative of I_f at x_0 on $\mathrm{L}_\phi(\Omega,\mathrm{E})$. □

Corollary 10.6 *Let ϕ be a Young integrand, f be a real valued integrand with a Fréchet-derivative $x^* \in \mathrm{E}_{\phi^*}(\Omega,\mathrm{E}^*)$ along $x_0 \in \mathrm{L}_\phi(\Omega,\mathrm{E})$. If in addition the condition $(\mathscr{S}_{\phi,x_0})$ holds, then I_f is Fréchet-differentiable at x_0 on $\mathrm{L}_\phi(\Omega,\mathrm{E})$. Moreover $\mathrm{I}'_f(x_0) = f'(x_0)$.*

Proof. Remark that if f satisfies $(\mathscr{S}_\phi, x_0)$, since for almost every $\omega \in \Omega$, the function f_ω is Lipschitzian on every ball of E, thus $\partial^{\mathrm{C}} f(e) = -\partial^{\mathrm{C}} - f(e)$, and the Young integrands being even then $-f$ satisfies $(\mathscr{S}_{\phi,x_0})$ too. Moreover due to Corollary 8.14, I_f is finite on a ball of $\mathrm{L}_\phi(\Omega,\mathrm{E})$ centered at x_0, thus locally $\mathrm{I}_f(x) = -\mathrm{I}_{-f}(x)$. Proposition 10.5 ensures that $f'(x_0) \in \partial^{\mathrm{F}}\mathrm{I}_f(x_0) \cap -\partial^{\mathrm{F}} - \mathrm{I}_f(x_0)$ this proves that I_f Fréchet-differentiable at $x_0 \in \mathrm{E}_\phi(\Omega,\mathrm{E})$ on $\mathrm{L}_\phi(\Omega,\mathrm{E})$ and $\mathrm{I}'_f(x_0) = f'(x_0)$. □

For clarity now we will restrict ourselves to the case of Lebesgue spaces. The following result is a permanence property.

Proposition 10.7 *Let $1 < p \leq \infty$. If the integrand f has the Fréchet-dlcp at $x_0 \in \mathrm{L}_p(\Omega,\mathrm{E})$ and $x^* \in \mathrm{L}_q(\Omega,\mathrm{E}^*_{\sigma^*})$, then the function $f - \langle x^*,\rangle$ has the Fréchet-dlcp at x_0.*

Proof of Proposition 10.7. For every measurable set A, and every $x \in \mathrm{L}_p(\Omega,\mathrm{E})$, $x^* \in \mathrm{L}_q(\Omega,\mathrm{E}^*)$, we have the upper bound:

$$\int_{\mathrm{A}} |\langle x^*, x\rangle| d\mu \leq \|x^* 1_{\mathrm{A}}\|_q \|x\|_p .$$

Since $1 \leq q < \infty$, for every norm bounded sequence $(x_n)_n$, the sequence $(\langle x^*, x_n\rangle)_n$ is uniformly integrable. Moreover setting $g = f - \langle x^*, .\rangle$, we have: $[g] = [f] - \langle x^*, .\rangle$ and since f has the Fréchet-dlcp at $x_0 \in \mathrm{X}$, the integrand g has the Fréchet-dlcp at x_0. □

Proposition 10.8 *Let $1 \leq p \leq \infty$. Let $x_0 \in \mathrm{L}_p(\Omega,\mathrm{E})$ with $f(x_0)$ integrable. Suppose that the integral functional I_f is Fréchet-subdifferentiable at x_0 with a Fréchet-subderivative $x_0^* \in \mathrm{L}_q(\Omega,\mathrm{E}^*_{\sigma^*})$. Then $f - \langle x_0^*, \rangle$ has the Fréchet-differential lower compactness property. Moreover if $p \neq 1$, for every $x^* \in \mathrm{L}_q(\Omega,\mathrm{E}^*_{\sigma^*})$ the integrand $f - \langle x^*, \rangle$ has the Fréchet-dlcp at x_0.*

Proof of Proposition 10.8. If I_f is Fréchet-subdifferentiable at x_0, with a Fréchet-subderivative $x_0^* \in \mathrm{L}_q(\Omega,\mathrm{E}^*_{\sigma^*})$ then Theorem 10.2 asserts that $f - \langle x_0^*, \rangle$ has the Fréchet-differential lower compactness property. Moreover if $p \neq 1$, due to Proposition 10.7 for every $x^* \in \mathrm{L}_q(\Omega,\mathrm{E}^*_{\sigma^*})$ the integrand $f - \langle x^*, \rangle$ has the Fréchet-dlcp at x_0. □

Theorem 10.9 *Suppose $1 < p \leq \infty$, if the integrand f has the Fréchet-dlcp at $x_0 \in \mathrm{L}_p(\Omega,\mathrm{E})$, then:*

$$\mathrm{L}_q(\partial^{\mathrm{F}} f(x_0))_{\sigma^*} \subseteq \partial^{\mathrm{F}} \mathrm{I}_f(x_0) .$$

Proof of Theorem 10.9. From Proposition 10.7, since $p \neq 1$, for every $x^* \in \mathrm{L}_q(\partial^{\mathrm{F}} f(x_0))_{\sigma^*}$ the integrand $f - \langle x^*, \rangle$ has the Fréchet-dlcp at x_0 and Theorem 10.2 ensures that x^* is a Fréchet subderivative of I_f at x_0. □

Corollary 10.10 *Suppose $1 < p \leq \infty$. If the integral functional I_f is Fréchet subdifferentiable at $x_0 \in \mathrm{L}_p(\Omega,\mathrm{E})$ with a Fréchet-subderivative $x_0^* \in \mathrm{L}_q(\Omega,\mathrm{E}^*_{\sigma^*})$, then the integrand f has the Fréchet-dlcp at x_0 and:*

$$\mathrm{L}_q(\partial^{\mathrm{F}} f(x_0))_{\sigma^*} \subseteq \partial^{\mathrm{F}} \mathrm{I}_f(x_0) .$$

Proof of Corollary 10.10. Due to Proposition 10.8 the integrand f has the Fréchet-dlcp at $x_0 \in \mathrm{L}_p(\Omega,\mathrm{E})$, and the result is a consequence of Theorem 10.9. □

Corollary 10.11 *Suppose $1 < p < \infty$. Given $x_0 \in \mathrm{L}_p(\Omega,\mathrm{E})$ and a measurable integrand $f : \Omega \times \mathrm{E} \to \mathbb{R}$ satisfying condition $(\mathscr{S}_p)$. Then*

$$\mathrm{L}_q(\partial^{\mathrm{F}} f(x_0))_{\sigma^*} \subset \partial^{\mathrm{F}} \mathrm{I}_f(x_0) .$$

Proof. It is an immediate consequence of Proposition 10.5. □

Corollary 10.12 *Suppose $p = \infty$, given $x_0 \in \mathrm{L}_\infty(\Omega,\mathrm{E})$ and a measurable integrand $f : \Omega \times \mathrm{E} \to \overline{\mathbb{R}}$ satisfying the condition $(\mathscr{S}_\infty)$. Then*

$$\mathrm{L}_1(\partial^{\mathrm{F}} f(x_0))_{\sigma^*} \subset \partial^{\mathrm{F}} \mathrm{I}_f(x_0) .$$

Proof. From Corollary 8.17 the integrand f has the Fréchet-dlcp, and from Proposition 10.7, for every element $y^* \in \mathrm{L}_1(\partial^{\mathrm{F}} f(x_0))_{\sigma^*}$, the integrand $f - \langle y^*, \rangle$ has the Fréchet-dlcp at the point x_0, and Theorem 10.2 allows to conclude. □

11 More on weak Hadamard subdifferentiability.

The author in [25], makes a first study of the strong Hadamard subdifferentiability of integral functionals on Lebesgue spaces, related with the properties of the differential quotients. In a initial version of this article, when E is reflexive, all the results on Fréchet-subdifferentiability of the section 10 has been proved with the results of section 6 by considering the weak topology on $L_p(\Omega, E)$ and the weak star topology when $p = \infty$. On reflexive spaces the Fréchet bornology coincide with the weak Hadamard bornology therefore the Fréchet subdifferential coincide with the weak Hadamard subdifferential. Notice that this last result can be extended (the proof is omitted) to the case where X has a predual Y:

Lemma 11.1 *Suppose that* Y *is a separable predual of the Banach* $(X, \|.\|)$ *and* $x_0 \in X$*, then with* $\sigma_Y = \sigma(X, Y)$*:*

$$\partial^F f(x_0) \cap Y = \partial^{\sigma_Y} f(x_0) \cap Y .$$

Moreover, if $1 < p \leq \infty$, when E is reflexive, due to the Proposition 8.18, Lemma 11.1, the results of the sections 9 and 10, Corollary 9.6, Corollary 9.8, Proposition 10.8, Theorem 10.9, Theorem 10.2 with its corollaries can be rephrased in terms of the weak dlcp (weak star if $p = \infty$) and of the weak (weak star if $p = \infty$) Hadamard subdifferentiability. Therefore the study of the weak (weak star) Dini Hadamard subdifferentiability is reduced to the study of Fréchet subdifferentiability in the cases $1 < p \leq \infty$. As a consequence, only the case $p = 1$ may be of interest. In the sequel, E is a separable reflexive space and we consider a $\mathbb{T} \otimes \mathscr{B}(E)$-measurable extended real valued integrand f and its differential quotient $[f]$. Corollary 9.8 gives a complete characterization of weak Hadamard subdifferentiability when the measure is atomless. The following statement is the analog of both results Theorem 10.9 and Corollary 10.10 and gives, with Theorem 11.3, practical sufficient conditions for the weak Hadamard subdifferentiability.

Theorem 11.2 *Let* $p = 1$*. If the integral functional* I_f *is weakly Hadamard subdifferentiable at* x_0 *on* $L_1(\Omega, E)$ *then the integrand* f *has the* σ*- dlcp at* x_0*. If the integrand* f *has the* σ*-dlcp at* x_0*, then* $L_\infty(\partial^\sigma f(x_0)) \subseteq \partial^\sigma I_f(x_0)$.

Proof of Theorem 11.2. Suppose that x^* is a weak Hadamard subderivative of I_f at x_0. Since the weak Hadamard bornology is solid, using Theorem 9.2 we deduce that the integrand $f - \langle x^*, . \rangle$ has the weak Hadamard dlcp; but due to the Dunford-Pettis criterion, for every relatively weakly compact sequence $(x_n)_n$ the sequence $(\langle x^*, x_n \rangle)_n$ is uniformly integrable, therefore f has the weak Hadamard dlcp. Conversely, let $(r_n)_n$ be a sequence of positive real numbers converging to the origin. Define $f_n(\omega, e) = [f]_\omega(x_0(\omega), e, r_n)$. For each $x^* \in L_\infty(\partial^\sigma f(x_0))$ we have for almost every $\omega \in \Omega$ the inequalities:

$$\langle x^*(\omega), . \rangle \leq f^\sigma_\omega(x_0(\omega), .)^{**} \leq (seq\, \sigma - li_e f_{n_\omega})^{**} .$$

Applying Corollary 6.5 to the sequence $(I_{f_n})_n$ we obtain for every $x \in L_1(\Omega, E)$,

$$I_{\langle x^*, x \rangle} \leq I_{f^\sigma(x_0, .)^{**}}(x) \leq I_{(seq\, \sigma - li_e f_n)^{**}}(x) \leq seq\, \sigma - li_e I_{f_n}(x) .$$

This proves that for each $x \in L_1(\Omega, E)$, $\int_\Omega \langle x^*, x \rangle d\mu \leq I^\sigma_f(x_0; x)$ or equivalently $x^* \in \partial^\sigma I_f(x_0)$. $\square$

Theorem 11.3 *Let $p = 1$, $x_0 \in L_1(\Omega, E)$, and consider the following assertions*
(a) *The integrand f has the σ-dlcp at x_0,*
(b) *There exists a positive constant c such that the integrand f satisfies out of a negligible set, for every $e \in E$,*

$$f(x_0 + e) \geq f(x_0) - c\|e\|.$$

Then $(b) \Rightarrow (a)$. If the measure μ is atomless these assertions are equivalent.

Proof of Theorem 11.3. By the Dunford-Pettis criterion, every sequentially weakly compact sequence $(x_n)_n$ is uniformly integrable. Therefore when (b) holds, for every sequence $(r_n)_n$ of positive real numbers converging to the origin, the sequence $([f]^-(x_0, r_n, x_n))_n$ is uniformly integrable. This proves that $(b) \Rightarrow (a)$. If the measure is atomless and (a) is true, then the integrand f has the (strong) Hadamard-dlcp at x_0, thus [25] Theorem 4.1 and [25] Corollary 4.2 or [51] Proposition 9, show that $(a) \Rightarrow (b)$. □

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References

[1] H. Attouch: *Variational Convergence for Functions and Operators.* Pitman. 1984.

[2] J.P. Aubin, H.Frankowska: *Set-Valued Analysis.* Systems & Control: Foundations & Applications 2. Birkhäuser Boston Inc Boston 1990.

[3] E. J. Balder: A general approach to lower semicontinuity and lower closure in optimal control theory. *SIAM. J. Control and Optimisation* **22** (1984) pp 570-598.

[4] E. J. Balder: Necessary and sufficient conditions for L_1-strong-weak lower semicontinuity of integral functionals. *Non linear Analysis* **11** (1987) pp 1399-1404.

[5] E. J. Balder: On seminormality of integral functionals and their integrands. *SIAM. J. Control and Optimisation* **24** *n*° 1 (1986) pp 95-120.

[6] A. Bourass, E. Giner: Kuhn-Tucker Conditions and Integral Functionals. *Journal of Convex Analysis* Vol **8** (2001) n^0 2 pp 533-553 .

[7] A. Bourass, B. Ferrahi, O. Kahlaoui: Semi-continuité inférieure d'une fonctionnelle intégrale. *Bulletin of the Greek Mathematical Society.* **47** (2003) pp 31-58.

[8] N. Bourbaki: *Espaces Vectoriels Topologiques.* Chapitres 1 à 5. Masson 1981.

[9] C. Castaing, M. Valadier: *Convex Analysis and Measurable Multifunctions.* Lecture Notes in Math. vol 580, Springer-Verlag Berlin and New York, 1977

[10] C. Castaing, P. Clauzure: Semi-continuité des fonctionnelles intégrales. *Acta Mathematica Vietnamica* **7** *n*° 2 (1982) pp 139-170.

[11] C. Castaing, P.R. de Fitte, M. Valadier: Young measures on topological spaces. *Mathematics and its Applications. Kluwer Academic Publishers.* **571** 2004.

[12] J. Couvreux: From the Mosco and slice convergences for convex normal integrands to that of integral functionals on L_p spaces. *Siam J. Control, Optim* **39** *n*° 1, pp 179-195.

[13] F.H. Clarke: *Optimization and Nonsmooth Analysis.* Canadian Mathematical Society Series of Monographs and Avanced Texts.Wiley Intersciences.Publ,Wiley, New York,1983.

[14] N. H. Chieu: The Fréchet and Limiting subdifferentials of integral functionals on the space $L_1(\Omega, E)$. *J. Mathematical Analysis and Applications.* **(54)**, *n*° 1 (1976) pp 230-244.

[15] G. Dal Maso: *An Introduction to* Γ*-convergence.* Birkhäuser, Boston 1993.

[16] C. Dellacherie, P. A. Meyer: *Probabilités et Potentiels* Hermann, Paris (1974).

[17] J. Diestel, J.J. Uhl: *Vector Measures* Math. Surveys Monographs, vol **15**, Amer. Math. Soc. Providence, RI, 1977.

[18] J. Diestel, W. M. Ruess, W. Chachermayer: Weak Compactness in $L^1(\mu, X)$. Proc. A.M.S. **118** (2) (1993) pp 447-453.

[19] N. Dunford, J. T. Schwartz: *Linear Operators 1. General Theory* John Wiley, New York, 1958.

[20] I. Ekeland, R. Temam: *Convex Analysis and Variational Problems.* North Holland, Amsterdam, 1974.

[21] I. Fonseca, G. Leoni: *Modern Methods in the Calculus of Variations:* L_p *spaces.* Springer 2007.

[22] A. Fougères: Coercivité des intégrandes convexes normales. Applications à la minimisation des fonctionnelles intégrales et du calcul des variations (I). *Seminaire D'analyse convexe, Montpellier,* vol 6 exposé **19** 1976.

[23] E. Giner: Décomposabilité et dualité des espaces d'Orlicz au sens de J. Chatelain et A. Kozeck. *C. R. Acad. Sc. Paris,* **t.284** 1977.

[24] E. Giner: Lipschitzian properties of integral functionals on Lebesgue spaces $L_p, 1 \le p < \infty$. *Set-Valued Analysis* **15** (2007) pp 125-138.

[25] E. Giner: Calmness properties and contingent subdifferential of an integral functional on Lebesgue spaces $L_p, 1 \le p < \infty$. *Set Valued Analysis.* **17** (2010) pp 223-243.

[26] E. Giner: Fatou's Lemma and strong lower epi-limits of integral functionals *J. of Mathematical Analysis and Applications* **394** n^0 1, 2012, pp 13-29).

[27] E. Giner: An upper bound for the convergence of integral functionals. *Journal of Convex Analysis.* (2013) **Vol.20** (2) pp 355-376.

[28] E. Giner: Some regularity properties of integral functionals (Submitted).

[29] E. Giner: Clarke and limiting subdifferentials of integral functionals. (to submit).

[30] E. Giner: Lower compactness properties for a sequence of integrands. (manuscript in preparation).

[31] F. Hiai, H. Umegaki: Integrals, conditional expectation and martingales of multivalued functions. *J. of Multivariate Analysis* **7** (1977) pp 149-182.

[32] R.B. Holmes: *Geometric Functional Analysis and its Applications*. Springer Verlag, Berlin 1975.

[33] G. A Hunt: *Martingales et Processus de Markov*. Monographie de la Société Mathématique de France **1** Dunod, Paris 1966.

[34] A-D. Ioffe: On lower semicontinuity of integral functionals (I) *SIAM. J. Control Optimization* **15** n^0 4, (1977) pp 521-538.

[35] A-D. Ioffe: On lower semicontinuity of integral functionals (II) *SIAM. J. Control Optimization* **15** n^0 6, (1977) pp 991-1000.

[36] A. and C. Ionescu-Tulcea *Topics in the Theory of Lifting* Ergeb. Math. Grenzgeb (3), Band 48, Springer-Verlag New York (1963).

[37] J. L. Joly: Une famille de topologies sur l'ensemble des fonctions convexes pour lesquelles la polarité est bicontinue. *J. Mathématiques pures et appliquées* **(52)**, $n°$ 9 (1973) pp 421-441.

[38] J. L. Joly, F. De Thélin: Convergence of convex integrals in L_p spaces. *Journal of Mathematical Analysis and Applications.* **(54)**, $n°$ 1 (1976) pp 230-244.

[39] C. M. Marle: *Mesures et Probabilités*. Hermann, Paris 1974.

[40] J. J. Moreau: *Fonctionnelles convexes*. Cours au Collège de France, 1967.

[41] U. Mosco: Convergence of convex sets and solutions of variational inequalities. *Adv. Math.* **3**, (1969) pp 510-585.

[42] U. Mosco: On the continuity of the Young-Fenchel transform. *J. Math. Anal.* **35**, (1971) pp 518-535.

[43] B. Mordukhovich: *Variational Analysis and Generalized Differentiation I.* Springer Verlag Berlin

[44] J. Neveu: *Bases Mathématiques du Calcul des Probabilités.* Masson, Paris (1964)

[45] C. Olech: Weak lower semicontinuity of integral functionals. *J. of Optimisation Theory and Applications* **19** $n°$ 1 (1976) pp 3-16.

[46] C. Olech, Weak lower semicontinuity of integral functionals. *J. Optim Theory and Appl.* **19**, *n*° 1 (1976) pp 135-142.

[47] C. Olech, A characterization of L_1-weak lower semicontinuity of integral functionals. *Bull. Acad. Polon. Sci.* **25**, (1977) pp 135-142.

[48] J.P. Penot, C. Zalinescu: Continuity of usual operations and variational convergences. *Set-Valued Analysis* **11** (2003) pp 225-256.

[49] J.P. Penot, C. Zalinescu: Continuity of the Legendre-Fenchel transform for some variational convergences. *Optimization* **53** (2004) pp 549-562.

[50] J.P. Penot, C. Zalinescu: Persistence and stability of solutions of Hamilton-Jacobi equations. *J. Mathematical Analysis and Applications* **347** (2008) pp 188-203.

[51] J.P. Penot: Image space approach and subdifferentials of integral functionals. *Optimization* **59** (8)(2010), pp 1-19.

[52] J.P. Penot: Calculus Without Derivatives, *Graduate Text in Maths, 266* Springer New York 2013.

[53] R. T. Rockafellar: Convex integral functionals and duality. in : *Contributions to nonlinear Functional Analysis,* edited by E. Zarantonello, Academic Press, New York-London (1971) pp 215-236.

[54] R.T. Rockafellar: Integral functionals, normal integrands and measurable selections. *Springer-Verlag Series Lect. Notes in Math*, **543**, (1976) pp 157-206 .

[55] R.T. Rockafellar, J.B. Wets: *Variational Analysis*. Springer, New York 1998 .

[56] A. Salvadori: On the M-convergence for integral functionals on $L_p(X)$. *Atti. Sem. Mat. Fis. Univ. Modena.* **33** (1984) pp 137-154.

[57] A. Salvadori: On the M-convergence for integral functionals on $L_p(X)$. *Séminaire d'Analyse Convexe de Montpellier* (1985) *n*° 5.

[58] R. L Schiling: *Measures, Integrals and Martingales*. Cambridge University Press, (2005).

[59] M. Slaby: Strong Convergence of Vector Valued Pramarts and Subpramarts. *Probability and Math. Stat.* **5** (1985) pp 187-196.

[60] M. Valadier: Convex integrands on Suslin locally convex spaces. *Pacific Journal of Mathematics.* **59** *n*° 1. pp 267-276.